Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

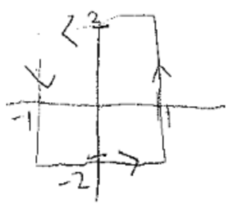
- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx.$
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-1 \le x \le 1\,,\ -2 \le y \le 2\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^8} + y^2\,,\ \sin(y^4) - 2x \right\rangle.$$

Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$



By Green's Theorem $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \text{curl}_z F \, dA$

$$= \int_{-1}^{1} \int_{-2}^{2} -2-2y \, dy\, dx$$

$$= \int_{-1}^{1} \left[-2y - y^2\right]_{-2}^{2} dx$$

$$= \int_{-1}^{1} (-4-4) - (-4+4) \, dx$$

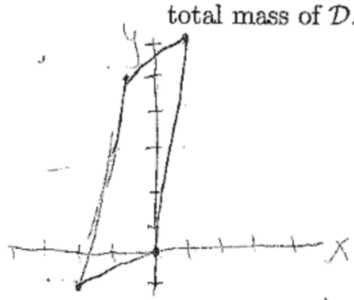
$$= \int_{-1}^{1} -8 \, dx$$

$$= \left[-8x\right]_{-1}^{1}$$

$$= -8 - (8)$$

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = -16}$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-2,-1), (0,0), (1,6), (-1,5)$ (where distance is measured in meters) and with mass density $\delta(x,y) = (2y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.



Total mass of $D = \iint_D \delta(x,y)\,dA$

With change of variables

$D = \{-1 \le u \le 0,\ 0 \le v \le 1\}$

$(x,y) = u\langle a,b\rangle + v\langle c,d\rangle$

$x = au + cv$ $\quad \langle a,b\rangle = \langle 2,1\rangle$

$y = bu + dv$ $\quad \langle c,d\rangle = \langle 1,6\rangle$

Change of variables

$x = 2u + v$

$y = u + 6v$

$u = \frac{x-v}{2}$

$u = y - 6v$

$y - 6v = \frac{x-v}{2}$

$2y - 12v = x - v$

$11v = 2y - x$

$\boxed{v = \frac{2y-x}{11}}$

$u = \dfrac{\frac{11x}{11} - \frac{2y-x}{11}}{2}$

$u = \frac{12x-2y}{22} = \boxed{\frac{6x-y}{11} = u}$

Compute the Jacobian: $\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{vmatrix}2 & 1\\ 1 & 6\end{vmatrix} = 11$

Then mass of $D = \int_{-1}^{0}\int_{0}^{1} (2y-x)\cdot 11\,dv\,du$

$= 121\int_{-1}^{0}\int_{0}^{1} \frac{2y-x}{11}\,dv\,du$

$= 121\int_{-1}^{0}\int_{0}^{1} v\,dv\,du$

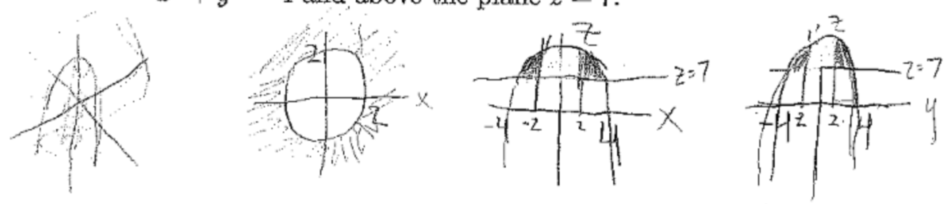
$= 121\int_{-1}^{0} \frac{1}{2}\,du$

$= 121\left(\frac{1}{2}\right)$

mass of $D = \frac{121}{2}$

The mass of lamina D is $\frac{121}{2}$ Kg

3. (12 points) Find the area of the part of the paraboloid $z = 16 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 4$ and above the plane $z = 7$.



Parametrize by $G(r,\theta) = \langle r\cos\theta, r\sin\theta, 16-r^2 \rangle$

Solving for boundary of r:

$7 = 16 - r^2$

$r^2 = 9$

$r = 3$

for $2 \le r \le 3$, $0 \le \theta \le 2\pi$

$\frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$

$\frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$\frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r^2 \rangle$

Take opposite normal: $\langle 2r^2\cos\theta, 2r^2\sin\theta, -r^2 \rangle$

$\text{Area} = \iint_S 1\, dS = \int_0^{2\pi}\int_2^3 \left\| \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right\| dr\, d\theta$

$\left\| \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^4}$

$= \sqrt{5r^4}$

$= r^2\sqrt{5}$

$= \int_0^{2\pi}\int_2^3 r^2\sqrt{5}\, dr\, d\theta$

$= \int_0^{2\pi} \left(\frac{27\sqrt{5}}{3} - \frac{8\sqrt{5}}{3} \right) d\theta$

$= 2\pi \left(\frac{19\sqrt{5}}{3} \right)$

Surface area $= \frac{38\pi\sqrt{5}}{3}$

The area of the surface is equal to $\frac{38\pi\sqrt{5}}{3}$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x, y)$, $g(x, y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_{\mathcal{C}} f\nabla g \cdot \mathbf{n}\, ds - \oint_{\mathcal{C}} g\nabla f \cdot \mathbf{n}\, ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

$$\iint_D f\Delta g\, dA = \iint_D f \operatorname{div}\nabla g\, dA$$
$$= \oint_C f\nabla g \cdot \hat{n}\, ds - \iint_D \nabla f\, \nabla g\, dA$$

$$\iint_D g\Delta f\, dA = \iint_D g \operatorname{div}\nabla f\, dA$$
$$= \oint_C g\nabla f \cdot \hat{n}\, ds - \iint_D \nabla f\, \nabla g\, dA$$

Then

$$\iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \oint_C f\nabla g \cdot \hat{n}\, ds - \cancel{\iint_D \nabla f\, \nabla g\, dA} - \left(\oint_C g\nabla f \cdot \hat{n}\, ds - \cancel{\iint_D \nabla f\, \nabla g\, dA}\right)$$

$$\iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \oint_C f\nabla g \cdot \hat{n}\, ds - \oint_C g\nabla f \cdot \hat{n}\, ds$$

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

$$\iint_D fg = \frac{1}{\lambda}\iint_D g\Delta f\, dA = \oint_C g\nabla f\cdot\hat{n}\,ds - \iint_D \nabla f\,\nabla g\, dA$$

$$\iint_D fg = \frac{1}{\mu}\iint_D f\Delta g\, dA = \oint_C f\nabla g\cdot\hat{n}\,ds - \iint_D \nabla f\,\nabla g\, dA$$