Math 32B Midterm 2D

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TOTAL POINTS

31 / 40

QUESTION 1

1 Green's Theorem 6 / 6

✓ + **2 pts** **Correct application of Green's Theorem**

✓ + **1 pts** **Correct computation of $$\mathrm{curl}_z\mathbf{F} = -2-2y$$**

✓ + **1 pts** **Correct limits for the rectangle (must have all four correct to receive credit)**

✓ + **1 pts** **Correct answer of $$-16$$ (requires correct integrand and limits to receive credit)**

✓ + **1 pts** **Solution clearly explained (and at the very least should mention Green's Theorem)**

+ **0 pts** No credit due

+ **1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 13 / 14

✓ + **2 pts** **Linear change of variables**

✓ + **3 pts** **Appropriate linear change of variables**

✓ + **2 pts** **Correct (u,v) region**

+ **1 pts** Correct Jacobian

✓ + **1 pts** **Use Jacobian**

✓ + **2 pts** **Correctly substitute u and v in δ/integrand.**

✓ + **1 pts** **Calculate correctly**

✓ + **1 pts** **Clear and organized solution, units**

✓ + **1 pts** **Accurate diagram, or accurate description of (x,y) region**

+ **1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))

+ **0 pts** No credit due

+ **1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

Used negative Jacobian.

QUESTION 3

3 Surface integral 4 / 12

+ **4 pts** Correct parametrization and parameter domain

+ **2 pts** Partial credits for parametrization

+ **4 pts** Correct tangent vector and normal vector calculation

✓ + **2 pts** **Partial credits for tangent and normal**

+ **4 pts** Correct double integral calculation

✓ + **2 pts** **Partial credits for double integral**

- **1 pts** You are almost there.

+ **1 pts** Almost nothing correct.

+ **0 pts** Nothing correct

QUESTION 4

4 Integration by parts 8 / 8

✓ + **1 pts** **Clear explanation**

✓ + **3 pts** **(a) correct**

✓ + **4 pts** **(b) correct**

+ **0 pts** Incorrect

+ **2 pts** (a) incomplete argument, but right idea

+ **2 pts** (b) incomplete argument, but right idea

+ **2 pts** (a) slight error

+ **3 pts** (b) slight error/unfinished

+ **1 pts** (a) started correctly, e.g. wrote integration by parts formula

+ **1 pts** (b) started correctly

gradescope

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: [redacted]
SID: [redacted]
TA Section: 1C

Time Limit: 50 Minutes

Version (↓)

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

$\iint_D f \,\mathrm{div}(F)\, dA = \oint_C f F \cdot \hat{n}\, ds - \iint_D \nabla f \cdot F\, dA$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$\cos(A-B) = \cos A \cos B + \sin A \sin B$

$\oint_C F \cdot dr = \iint_D \mathrm{curl}_z F\, dA = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$

$\oint_C F \cdot \hat{n}\, ds = \iint_D \mathrm{div}\, F\, dA = \iint_D \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}\, dA$

Scalar $\|N(u,v)\|$

vector surface $\cdot N(u,v)$

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$

Probability formulas

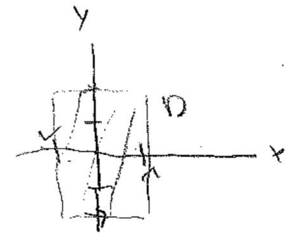
- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\, dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\, dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\, p_X(x)\, dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x, y)$ then
  - The total probability $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{X,Y}(x, y)\, dx dy = 1$
  - The probability that $(X, Y) \in \mathcal{D}$ is $\mathbb{P}[(X, Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x, y)\, dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X, Y)$ is $\mathbb{E}[f(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, p_{X,Y}(x, y)\, dx dy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-1 \le x \le 1\,,\ -2 \le y \le 2\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^8} + y^2\,,\ \sin(y^4) - 2x \right\rangle.$$

Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$



using Green's Theorem,

$\oint_C F \cdot dr = \iint_D \text{curl}_z F \, dA$

Finding $\text{curl}_z F = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$, we compute

$\frac{\partial F_2}{\partial x} = -2$, $\frac{\partial F_1}{\partial y} = 2y$, Thus $\text{curl}_z F = -2 - 2y$

plugging this in $\oint_C F \cdot dr = \iint_D \text{curl}_z F \, dA = \iint_D -2-2y \, dA$.

we then put in the limits given and compute the double integral.

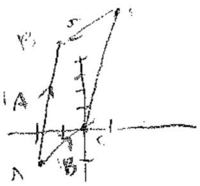
$\int_{-2}^{2}\int_{-1}^{1} -2-2y \, dx\,dy = \int_{-2}^{2} -2x - 2xy \Big|_{-1}^{1} dy = \int_{-2}^{2} -2-2y - (2+2y)\, dy$

$= \int_{-2}^{2} -4 - 4y \, dy = -4y - 2y^2 \Big|_{-2}^{2} = -8 - 8 - (8 - 8)$

$= -16$

Thus our answer is -16.

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-2,-1),(0,0),(1,6),(-1,5)$ (where distance is measured in meters) and with mass density $\delta(x,y)=(2y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.



we compute vectors A and B.

$A = \langle -1+2, 5+1 \rangle = \langle 1, 6 \rangle$

$B = \langle 0+2, 0+1 \rangle = \langle 2, 1 \rangle$

we know that $x = Au + Cv$, $y = Bu + Dv$

thus $x = u + 2v$ and $y = 6u + v$

solving for u, v in terms of x, y we get

$u = x - 2v \rightarrow y = 6(x-2v) + v$ $\quad y = 6x - 12v + v$

$y - 6x = -11v$

$v = -\frac{1}{11}(y - 6x)$

$v = y - 6u \rightarrow x = u + 2(y - 6u) \rightarrow x = u + 2y - 12u$

$-2y + x = -11u$

$-\frac{1}{11}(x - 2y) = u$

solving for u, v boundaries we get

$-\frac{1}{11}(-2+2) = 0$ $\quad -\frac{1}{11}(1-12) = 1$ $\quad \leftarrow$ for u bounds $\quad 0 \le u \le 1$

$-\frac{1}{11}(-1+12) = -1$ $\quad -\frac{1}{11}(6-6) = 0$ $\quad \leftarrow$ for v bounds $\quad -1 \le v \le 0$

computing our Jacobian (G) we get $\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 6 & 1 \end{vmatrix}$ $\quad 1 - 12 = -11$

plugging this all into our double integral we obtain

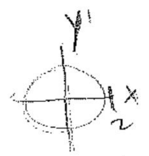
$-11 \int_{-1}^{0}\int_{0}^{1} 2(6u+v) - (u+2v)\, du\, dv$

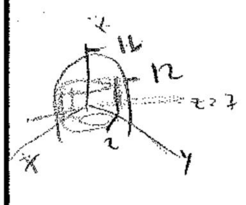
$= -11 \int_{-1}^{0}\int_{0}^{1} 12u + 2v - u - 2v\, du\, dv = -11 \int_{-1}^{0}\int_{0}^{1} 11u\, du\, dv$

$= -11 \int_{-1}^{0} \frac{11}{2}u^2 \Big|_0^1 dv = -11 \int_{-1}^{0} \frac{11}{2} dv = -11 \left(\frac{11}{2} v \Big|_{-1}^{0} \right) = \left(\frac{11}{2}\right) - 11$

$= -\frac{121}{2}$ g

3. (12 points) Find the area of the part of the paraboloid $z = 16 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 4$ and above the plane $z = 7$.





we find the intersection of $z = 16 - x^2 - y^2$

and $x^2 + y^2 = 4$.

$16 - z = x^2 + y^2$ so we subsitute that

into $x^2 + y^2 = 4$, $16 - z = 4$, $z = 12$

we now parametrize our curve in terms of

z, θ

$G(z,\theta) = \langle 2\cos\theta, 2\sin\theta, z \rangle$

Finding the tangent vectors we get

$\frac{\partial G}{\partial z} = \langle 0, 0, 1 \rangle$ $\frac{\partial G}{\partial \theta} = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

Thus our normal $= \frac{\partial G}{\partial z} \times \frac{\partial G}{\partial \theta}$

i	j	k	i	j
0	0	1	0	0
$-2\sin\theta$	$2\cos\theta$	0	$-2\sin\theta$	$2\cos\theta$

$-2\sin\theta\,\vec{j} - 2\cos\theta\,\vec{i}$

$N(z,\theta) = \langle -2\cos\theta, -2\sin\theta, 0 \rangle$

Thus $\|N(z,\theta)\| = \sqrt{4} = 2$

Computing our area we set up and evaluate the double integral.

$$\int_0^{2\pi}\int_7^{12} 2\,dz\,d\theta = \int_0^{2\pi} 2z \Big|_7^{12} d\theta = \int_0^{2\pi} 2(12-7)\,d\theta = \int_0^{2\pi} 10\,d\theta$$

$$= 10\theta \Big|_0^{2\pi} = 20\pi \text{ units}^2$$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x,y)$, $g(x,y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_{\mathcal{C}} f\nabla g \cdot \mathbf{n}\, ds - \oint_{\mathcal{C}} g\nabla f \cdot \mathbf{n}\, ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

we recall the integration by parts formula is

$$\iint_D f \operatorname{div}(\vec{F})\, dA = \oint_C f\vec{F} \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \vec{F}\, dA$$

Substituting our scalar fields and vector fields respectively we obtain.

$$\iint_D f \operatorname{div}(\nabla g)\, dA - \iint_D g \operatorname{div}(\nabla f)\, dA = \oint_C f\nabla g \cdot n\, ds - \oint g\nabla f \cdot n\, ds$$

↑ Let $\vec{F} = \nabla g$ ↑ Let $\vec{F} = \nabla f$

$$\oint_C f(\nabla g) \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA - \left(\oint_C g(\nabla f) \cdot \hat{n} - \iint_D \nabla g \cdot \nabla f\, dA\right)$$

simplifying, we get

$$\oint_C f(\nabla g) \cdot \hat{n}\, ds - \iint_D \nabla f \cdot \nabla g\, dA - \oint_C g(\nabla f) \cdot \hat{n} + \iint_D \nabla g \cdot \nabla f\, dA$$

$$= \oint_C f(\nabla g) \cdot \hat{n}\, ds - \oint_C g(\nabla f) \cdot \hat{n}$$ (since dot product is commutative)

□

which proves the identity.

(b) Suppose that $f(x, y)$, $g(x, y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x, y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x, y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x, y)$, $g(x, y)$ satisfy the boundary condition

$$f(x, y) = 0 \quad \text{for all } (x, y) \in \mathcal{C},$$
$$g(x, y) = 0 \quad \text{for all } (x, y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x, y) g(x, y)\, dA = 0.$$

substituting into the equation we get

$\iint_D f(\mu g)\, dA - \iint_D g(\lambda f)\, dA = \oint f \nabla g \cdot n\, ds - \oint_C g \nabla f \cdot n\, ds$

however $f(x,y) = 0$ and $g(x,y) = 0$ for all $(x,y) \in C$

thus

$\iint_D f(\mu g)\, dA - \iint_D g(\lambda f)\, dA = 0$

$\iint_D \mu f g\, dA - \iint_D \lambda f g\, dA = 0$

Thus when $\lambda \neq \mu$, $\iint_D f(x,y)\, g(x,y)\, dA = 0$