Math 32B Midterm 2U

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TOTAL POINTS

32 / 40

QUESTION 1

1 Green's Theorem 4 / 6

✓ **+ 2 pts Correct application of Green's Theorem**

+ **1 pts** Correct computation of $$\mathrm{curl}_z\mathbf{F} = 6xy+4$$

✓ **+ 1 pts Correct limits for the rectangle (must have all four correct to receive credit)**

+ **1 pts** Correct answer of $$32$$ (requires correct integrand and limits to receive credit)

✓ **+ 1 pts Solution clearly explained (and at the very least should mention Green's Theorem)**

+ **0 pts** No credit due

+ **1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 12 / 14

✓ **+ 2 pts Linear change of variables**

✓ **+ 3 pts Appropriate linear change of variables**

+ **2 pts** Correct (u,v) region

✓ **+ 1 pts Correct Jacobian**

✓ **+ 1 pts Use Jacobian**

✓ **+ 2 pts Correctly substitute u and v in δ/integrand.**

✓ **+ 1 pts Calculate correctly**

✓ **+ 1 pts Clear and organized solution, units**

✓ **+ 1 pts Accurate diagram, or accurate description of (x,y) region**

+ **1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))

+ **0 pts** No credit due

+ **1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

Very nicely written!

QUESTION 3

3 Surface integral 12 / 12

✓ **+ 4 pts Correct parametrization and domain**

+ **2 pts** Partial credits on parametrization

✓ **+ 4 pts Correct tangent and normal vector**

+ **2 pts** Partial credits on tangent and normal

✓ **+ 4 pts Correct double integral calculation**

+ **2 pts** Partial credits on double integral

- **1 pts** Almost there

+ **1 pts** Almost nothing correct

+ **0 pts** Nothing correct

QUESTION 4

4 Integration by parts 4 / 8

✓ **+ 1 pts Clear explanation**

✓ **+ 3 pts (a) correct**

+ **4 pts** (b) correct

+ **0 pts** Incorrect

+ **2 pts** (a) incomplete argument, but right idea

+ **2 pts** (b) incomplete argument, but right idea

+ **2 pts** (a) slight error

+ **3 pts** (b) slight error/unfinished

+ **1 pts** (a) started correctly, e.g. wrote integration by parts formula

+ **1 pts** (b) started correctly

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: [redacted]
SID:
TA Section:

Time Limit: 50 Minutes

Version $\textcircled{\uparrow}$

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \le x \le 2\,,\ -1 \le y \le 1\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y\,,\ \sin(e^y) + 3x^2y \right\rangle.$$

Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

We wish to evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where C is the boundary of the rectangle $D = \{-2 \le x \le 2, -1 \le y \le 1\}$ oriented CCW. Since C is CCW, the region is to our "left" as we walk along the curve.

By Green's theorem:

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA.$$

We compute $\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left[\sin(e^y) + 3x^2y\right]$

$$= 6xy$$

We compute $\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left[e^{x^3-x} - 4y\right]$

$$= -4$$

Therefore, $\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \left(6xy \boxed{-4}\right)$.

The integral can now be written as:

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{-2}^{2}\int_{-1}^{1}\left[6xy - 4\right] dy\,dx.$$

$$= \int_{-2}^{2}\left[3xy^2 - 4y\right]_{-1}^{1} dx$$

$$= \int_{-2}^{2}\left[3x(1)^2 - 4(1) - 3x(-1)^2 + 4(-1)\right] dx$$

$$= \int_{-2}^{2}\left[3x - 4 - 3x - 4\right] dx$$

$$= \int_{-2}^{2}\left[-8\right] dx$$

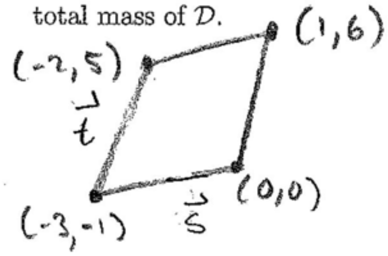
$$= -8\int_{-2}^{2} dx$$

$$= -8\left[2 - -2\right]$$

$$= -8(4) = 32$$

We have thus proved that $\oint_C \vec{F} \cdot d\vec{r} = 32$ where C is the boundary of the rectangle $D = \{-2 \leq x \leq 2, -1 \leq y \leq 1\}$, oriented counterclockwise.

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1), (0,0), (1,6), (-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y) = \frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.



We wish to find the total mass of the parallelogram with corners $(-3,-1)$, $(0,0)$, $(1,6)$, and $(-2,5)$ in kg.

Let $\vec{s} = \langle 3,1 \rangle$ and let $\vec{t} = \langle 1,6 \rangle$.

We can define a linear map G such that $G(u,v) = (3u+v,\ u+6v)$. So, $x = 3u+v$ and $y = u+6v$.

The Jacobian, Jac(G), is computed:

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} = 18 - 1 = 17.$$

The mass of D can be defined as the following integral:

$$M = \iint_D \delta(x,y)\, dA.$$

Since we changed δ to be in terms of u & v:

$$M = \iint_{\bar{D}} \delta(u,v)\, \|\mathrm{Jac}(G)\|\, du\, dv$$

where $\bar{D}$ is the domain of u and v.

$\bar{D}$ is the rectangle $[0,1] \times [0,1]$ since $G(0,1) = (1,6)$ and $G(1,0) = (3,1)$.

The mass is now the integral:

$$M = \int_0^1 \int_0^1 \cancel{\tfrac{1}{17}}(3y-x)(\cancel{17})\, du\, dv$$

$$M = \int_0^1 \int_0^1 [3(u+6v) - 3u - v]\, du\, dv$$

$$M = \int_0^1 \int_0^1 [\cancel{3u} + 18v - \cancel{3u} - v]\, du\, dv$$

$$M = \int_0^1 \int_0^1 17v\, du\, dv$$

$$M = \int_0^1 \int_0^1 17v \, dw \, dv.$$

$$M = 17 \int_0^1 \int_0^1 v \, dw \, dv$$

$$M = 17 \left[\int_0^1 v dv \right] \left[\int_0^1 dw \right]$$

$$M = 17 \left[\tfrac{1}{2} v^2 \Big|_0^1 \right] \left[1 - 0 \right]$$

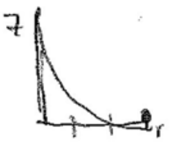
$$M = 17 \left[\tfrac{1}{2} - 0 \right] \left[1 \right]$$

$$M = \frac{17}{2} \text{ kg.}$$

The total mass of the lamina on the region $D = \frac{17}{2}$ kg.

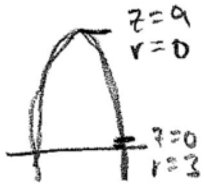
3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.

We wish to find the surface area of the paraboloid $z = 9 - x^2 - y^2$ that is outside $x^2 + y^2 = 1$ and above $z = 5$.

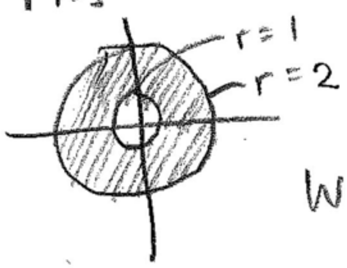


The surface area is defined as:

$$\iint_S dS = \iint_D \|N(u,v)\|\, du\, dv$$



The projection of $z = 9 - x^2 - y^2$ onto the xy plane is the equation $x^2 + y^2 = 9$.



However, at $z = 5$, the equation $z = 9 - x^2 - y^2$ becomes $x^2 + y^2 = 4$.

We parameterize the surface area:

$$G(r,\theta) = (r\cos\theta,\ r\sin\theta,\ r^2 - 9)$$

where $1 \le r \le 2$ and $0 \le \theta \le 2\pi$.

$$\frac{\partial G}{\partial r} = (\cos\theta, \sin\theta, 2r)$$

$$\frac{\partial G}{\partial \theta} = (-r\sin\theta, r\cos\theta, 0)$$

$$N(r,\theta) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \begin{vmatrix} \sin\theta & 2r \\ r\cos\theta & 0 \end{vmatrix}\hat{i} - \begin{vmatrix} \cos\theta & 2r \\ -r\sin\theta & 0 \end{vmatrix}\hat{j} + \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix}\hat{k}$$

$$= (0\cancel{\sin\theta} - 2r^2\cos\theta)\hat{i} - (0\cancel{\cos\theta} + 2r^2\sin\theta) + (r\cos^2\theta + r\sin^2\theta)$$

$$N(r,\theta) = \langle -2r^2\cos\theta,\ 2r^2\sin\theta,\ r \rangle$$

The surface integral is defined as:

$$\int_0^{2\pi} \int_1^2 \|N(r,\theta)\| \, dr \, d\theta$$

We compute $\|N(r,\theta)\|$:

$$N(r,\theta) = \langle -2r^2\cos\theta, + 2r^2\sin\theta, r \rangle$$

$$\|N(r,\theta)\| = \sqrt{(-2r^2\cos\theta)^2 + (2r^2\sin\theta)^2 + r^2}$$

$$= \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2}$$

$$= \sqrt{4r^4 + r^2}$$

$$= r\sqrt{4r^2+1}.$$

The integral is now:

$$= \int_0^{2\pi} \int_1^2 r\sqrt{4r^2+1} \, dr \, d\theta$$

let $u = 4r^2+1 \quad du = 8r\,dr$

$$= \frac{1}{8} \int_0^{2\pi} \int_1^2 u^{\frac{1}{2}} \, du \, d\theta$$

$$= \frac{1}{8} \int_0^{2\pi} \frac{2}{3}(4r^2+1)^{\frac{3}{2}} \Big|_1^2 \, d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} [4(2)^2+1]^{3/2} - [4(1)^2+1]^{3/2} \, d\theta$$

$$= \frac{1}{12} \int_0^{2\pi} [17^{3/2} - 5^{3/2}] \, d\theta$$

$$= \frac{1}{12} \left[17^{\frac{3}{2}}\theta - 5^{\frac{3}{2}}\theta \right]_0^{2\pi}$$

$$= \frac{1}{12} \left[17^{\frac{3}{2}}(2\pi) - 5^{\frac{3}{2}}(2\pi) - \cancel{17^{\frac{3}{2}}(0)} + \cancel{5^{\frac{3}{2}}(0)} \right]$$

$$= \frac{1}{12} \left[17^{\frac{3}{2}}(2\pi) - 5^{\frac{3}{2}}(2\pi) \right]$$

$$= \frac{\pi}{6} \left[17^{\frac{3}{2}} - 5^{\frac{3}{2}} \right]$$

The area of the given figure is $\frac{\pi}{6}\left[17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right]$.

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x,y)$, $g(x,y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\,dA - \iint_{\mathcal{D}} g\Delta f\,dA = \oint_{\mathcal{C}} f\nabla g\cdot\mathbf{n}\,ds - \oint_{\mathcal{C}} g\nabla f\cdot\mathbf{n}\,ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div}\nabla f$)

The general formula for integration by parts is:

$$\iint_D f \operatorname{div}\vec{F}\,dA = \oint_C (fF\cdot n)\,ds - \iint_D \nabla f\cdot F\,dA.$$

We compute $\iint_D f\Delta g\,dA$:

Let $\Delta g = \operatorname{div}(\nabla g)$, so:

$$\iint_D f\Delta g\,dA = \oint_C f\nabla g\cdot n\,ds - \iint_D \nabla f\nabla g\,dA$$

We compute $\iint_D g\Delta f\,dA$:

Let $\Delta f = \operatorname{div}(\nabla f)$, so:

$$\iint_D g\Delta f\,dA = \oint_C g\nabla f\cdot n\,ds - \iint_D \nabla g\nabla f\,dA.$$

We now compute:

$$\iint f\Delta g\,dA - \iint_D g\Delta f\,dA$$

$$\left[\oint_C f\nabla g\cdot n\,ds - \cancel{\iint_D \nabla f\nabla g\,dA}\right] - \left[\oint_C g\nabla f\cdot n\,ds - \cancel{\iint_D \nabla g\nabla f\,dA}\right]$$

Since $\iint_D \nabla f\nabla g\,dA = \iint_D \nabla g\nabla f\,dA$, the difference becomes:

$$\oint_C f\nabla g\cdot n\,ds - \oint_C g\nabla f\cdot n\,ds$$

Which is what we wished to prove. QED

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\, dA = 0.$$

Since Δf =