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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \le x \le 2\,,\ -1 \le y \le 1\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y\,,\ \sin(e^y) + 3x^2y \right\rangle.$$

Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

By Green's theorem, this is equal to

$$\iint_A \mathrm{curl}_z \vec{F}\, dA$$

we compute $\mathrm{curl}_z \vec{F}$

$$= \det \begin{vmatrix} \frac{\partial}{\partial x} & \partial/\partial y \\ e^{x^3-x} - 4y & \sin(e^y) + 3x^2y \end{vmatrix} = 6xy + 4$$

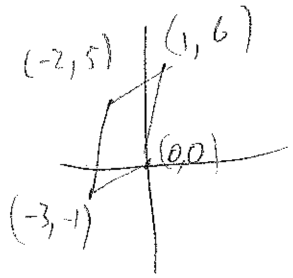
$$= \int_{-2}^{2}\int_{-1}^{1} 6xy + 4\, dy\, dx$$

$$= \int_{-2}^{2} 3xy^2 + 4y \Big|_{-1}^{1} dx$$

$$= \int_{-2}^{2} \cancel{3x} + 4 - \cancel{3x} + 4\, dx = \int_{-2}^{2} 8\, dx$$

$$= 32$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1),(0,0),(1,6),(-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y)=\frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.



The vectors are given by $\langle 3,1\rangle$ and $\langle 1,6\rangle$

$$\begin{cases} x = 3u+v \\ y = u+6v \end{cases}$$ Solving, we have

$v = x-3u$

$y = u+6x-18u$

$y-6x = -17u,\quad u = \frac{-y+6x}{17}$

$$v = x-3\left(\frac{-y+6x}{17}\right)$$
$$= x-\frac{-3y+18x}{17}$$
$$= \frac{17x}{17}+\frac{3y-18x}{17}$$
$$= \frac{3y-x}{17}$$

u goes from

$\frac{-17}{17} \le u \le 0$

$0 \le v \le 1$

the Jacobian is $\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 3 & 1 \\ 1 & 6 \end{vmatrix} = 17$

$\delta(u,v) = v$

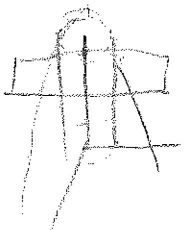
So, $M_D = \int_{-1}^{0}\int_{0}^{1} 17v \, dv \, du$

$= \int_{-1}^{0} \frac{17v^2}{2}\Big|_0^1 \, du$

$= \int_{-1}^{0} \frac{17}{2} \, du = \frac{17}{2} \text{ kg}$

3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.

The cylinder and paraboloid will intersect when they are equal.



$z = 9 - (x^2 + y^2)$, and $x^2 + y^2 = 1$ on the cylinder,

$z = 9 - 1 = 8$, so $r = 1$

The paraboloid can be parameterized by

$G(x,y) = \langle x, y, 9 - x^2 - y^2 \rangle$

$\text{Area}(S) = \iint_D \|n\| \, dA$

$\|n\| = \left\| \frac{\partial g}{\partial x} \times \frac{\partial g}{\partial y} \right\| = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix}$

$= \langle 2x, 2y, 1 \rangle$

$\|n\| = \sqrt{4x^2 + 4y^2 + 1} = \sqrt{4r^2 + 1}$

switching to polar, where $0 \leq \theta \leq 2\pi$, $1 \leq r \leq 2$

$5 = 9 - x^2 - y^2$

$-4 = -x^2 - y^2$, $r = 2$

$= \int_1^2 \int_0^{2\pi} r\sqrt{4r^2+1} \, d\theta \, dr$

$= \int_1^2 2\pi r \sqrt{4r^2+1} \, dr$

Let $u = 4r^2 + 1$

$du = 8r \, dr$

$\frac{1}{8} du = r \, dr$

$= \frac{\pi}{4} \int_5^{17} \sqrt{u} \, du$

back →

$$= \pi/4 \left(\frac{2u^{3/2}}{3} \right) \Big|_{u=5}^{17}$$

$$= \pi/6 \left(17^{3/2} - 5^{3/2} \right)$$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x, y)$, $g(x, y)$ we have the identity

$$\iint_{\mathcal{D}} f \Delta g \, dA - \iint_{\mathcal{D}} g \Delta f \, dA = \oint_{\mathcal{C}} f \nabla g \cdot \mathbf{n} \, ds - \oint_{\mathcal{C}} g \nabla f \cdot \mathbf{n} \, ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

we know that the by parts formula states

$$\iint_D f \operatorname{div} \vec{F} = \oint_C f \vec{F} \cdot \hat{n} \, ds - \iint_D F \cdot \nabla f \, dA$$

so, we know

$$\iint_D \underbrace{f \Delta g \, dA}_{\substack{\text{where } \vec{F} = \nabla g \\ f = f \\ \operatorname{div} F = \Delta g}} - \iint \underbrace{g \Delta f}_{\substack{\vec{F} = \nabla f \\ f = g \\ \operatorname{div} F = \Delta f}} = \oint f \nabla g \cdot n \, ds - \oint g \nabla f \cdot n \, ds$$

$$\cancel{- \iint \nabla f \nabla g \, dA + \iint \nabla g \nabla f \, dA}$$

The last two terms cancel, so

$$\iint_D f \Delta g \, dA - \iint g \Delta f = \oint f \nabla g \cdot n \, ds - \oint g \nabla f \cdot n \, ds$$

$\square$

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\, dA = 0.$$

we can bring the integrals together, so

$$\iint_D f\Delta g - g\Delta f \, dA = \oint_C \cancelto{0}{f} \nabla g \cdot n \, ds - \oint_C \cancelto{0}{g} \nabla f \cdot n \, ds$$

substituting:

$$\iint_D f\mu g - g\lambda f \, dA = 0$$

factoring out f and g, we have

$$\iint_D fg(\mu - \lambda)\, dA = 0$$

since $(\mu - \lambda)$ is constant, and are not equal,

$$(\mu - \lambda)\iint_D f(x,y)\, g(x,y)\, dA = 0$$

$$\iint_D f(x,y)\, g(x,y)\, dA = \frac{0}{(\mu - \lambda)} = 0$$

$\square$