Math 32B Midterm 2U

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TOTAL POINTS

36 / 40

QUESTION 1

1 Green's Theorem 6 / 6

- ✓ **+ 2 pts** **Correct application of Green's Theorem**
- ✓ **+ 1 pts** **Correct computation of $$\mathrm{curl}_z\mathbf{F} = 6xy+4$$**
- ✓ **+ 1 pts** **Correct limits for the rectangle (must have all four correct to receive credit)**
- ✓ **+ 1 pts** **Correct answer of $$32$$ (requires correct integrand and limits to receive credit)**
- ✓ **+ 1 pts** **Solution clearly explained (and at the very least should mention Green's Theorem)**
- **+ 0 pts** No credit due
- **+ 1 pts** Applied Green's Theorem correctly but with wrong orientation. (Partial credit)

QUESTION 2

2 Change of variables 14 / 14

- ✓ **+ 2 pts** **Linear change of variables**
- ✓ **+ 3 pts** **Appropriate linear change of variables**
- ✓ **+ 2 pts** **Correct (u,v) region**
- ✓ **+ 1 pts** **Correct Jacobian**
- ✓ **+ 1 pts** **Use Jacobian**
- ✓ **+ 2 pts** **Correctly substitute u and v in δ/integrand.**
- ✓ **+ 1 pts** **Calculate correctly**
- ✓ **+ 1 pts** **Clear and organized solution, units**
- ✓ **+ 1 pts** **Accurate diagram, or accurate description of (x,y) region**
- **+ 1 pts** Partial credit for error in finding (u,v) region, Jacobian, or δ(x(u,v),y(u,v))
- **+ 0 pts** No credit due
- **+ 1 pts** Sanity check: recognize that a negative answer is incorrect (does not apply if "corrected" by taking absolute value)

QUESTION 3

3 Surface integral 12 / 12

- ✓ **+ 4 pts** **Correct parametrization and domain**
- **+ 2 pts** Partial credits on parametrization
- ✓ **+ 4 pts** **Correct tangent and normal vector**
- **+ 2 pts** Partial credits on tangent and normal
- ✓ **+ 4 pts** **Correct double integral calculation**
- **+ 2 pts** Partial credits on double integral
- **- 1 pts** Almost there
- **+ 1 pts** Almost nothing correct
- **+ 0 pts** Nothing correct

QUESTION 4

4 Integration by parts 4 / 8

- ✓ **+ 1 pts** **Clear explanation**
- **+ 3 pts** (a) correct
- **+ 4 pts** (b) correct
- **+ 0 pts** Incorrect
- **+ 2 pts** (a) incomplete argument, but right idea
- **+ 2 pts** (b) incomplete argument, but right idea
- ✓ **+ 2 pts** **(a) slight error**
- **+ 3 pts** (b) slight error/unfinished
- **+ 1 pts** (a) started correctly, e.g. wrote integration by parts formula
- ✓ **+ 1 pts** **(b) started correctly**

(a) need dot product; (b) should have fg in both integrals, otherwise doesn't work

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: [redacted]
SID: [redacted]
TA Section: [redacted]

Time Limit: 50 Minutes

Version (↑)

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

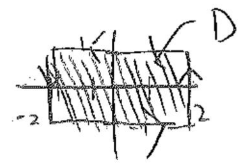
- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \le x \le 2\ ,\ -1 \le y \le 1\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y\ ,\ \sin(e^y) + 3x^2y \right\rangle.$$



Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

The curve is already oriented counter clockwise which hints that we can use Green's Theorem to solve this vector line integral. By Green's theorem, for an oriented curve C; $\iint_D \text{curl}_z \vec{F}\, dA = \int_C \vec{F} \cdot d\vec{r}$

We compute $\text{Curl}_z \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{\partial(\sin(e^y) + 3x^2y)}{\partial x} - \frac{\partial(e^{x^3-x} - 4y)}{\partial y} = 6xy - (-4)$

Therefore, $\text{curl}_z \vec{F} = 6xy + 4$.

Now, we apply Green's Theorem to compute the line integral:

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} \int_{-2}^{2} (6xy + 4)\, dx\, dy = \int_{-1}^{1} (3x^2y + 4x \Big]_{-2}^{2} dy = \int_{-1}^{1} \Big[3(2)^2 y + 4(2) - (3(-2)^2 y + 4(-2)\Big] dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_{-1}^{1} (12y + 8 - 12y + 8)\, dy = \int_{-1}^{1} 16\, dy = 16y \Big]_{-1}^{1} = 16(1) - 16(-1) = \boxed{32}.$$

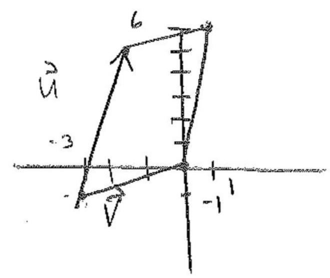
Therefore, by applying the vector form of Green's Theorem we were able to compute the vector line integral.

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1),(0,0),(1,6),(-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y)=\frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$. Total mass $= \iint_D \delta(x,y)\,dA$

First, lets sketch the domain to be able to detelmine a proper set of coordinates (u,v) to compute the integral in.

$\vec{u} = \langle -2-(-3), 5-(-1)\rangle = \langle 1, 6\rangle = \langle a, b\rangle$

$\vec{v} = \langle 0-(-3), 0-(-1)\rangle = \langle 3, 1\rangle = \langle c, d\rangle$



$X(u,v) = au + cv = u + 3v$

$Y(u,v) = bu + dv = 6u + v$

To be able to properly apply a change of variable to coordinates (u,v), we need to compute the Jacobian.

$$\mathrm{Jac}(G(u,v)) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \det\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 6 & 1\end{vmatrix} = 1 - 6(3) = -17.$$

We also need to solve for our domain in terms of (u,v), so we need to find (u,v) in terms of x and y.

$-6(x) = -6(u+3v)$
$y = 6u + v$

$y - 6x = -6u + 6u - 18v + v$
$17v = 6x - y$
$\boxed{v = \frac{1}{17}(6x - y)}$

$x = u + 3v$
$-3(y) = -3(6u+v)$

$x - 3y = u - 18u + 3v - 3v$
$17u = 3y - x$
$\boxed{u = \frac{1}{17}(3y - x)}$

Now, to find our domain in (u,v) we plug in the points $(-3,-1)$ and $(1,6)$.

$v(-3,-1) \le v \le v(1,6)$; $u(-3,-1) \le u \le u(1,6)$

$\frac{1}{17}(-18+1) \le v \le \frac{1}{17}(6-6)$; $\frac{1}{17}(3-3) \le u \le \frac{1}{17}(18-1)$

$-1 \le v \le 0$; $0 \le u \le 1$

Therefore our domain is $D;\ \{0 \le u \le 1,\ -1 \le v \le 0\}$

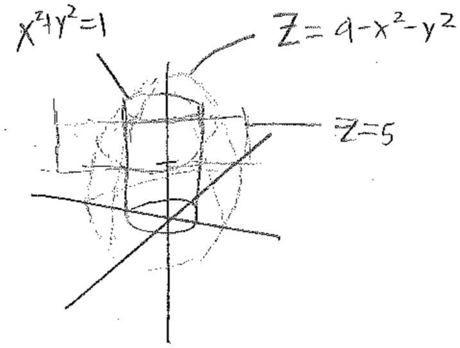
In terms of u and v, $\delta(x,y) = \frac{1}{17}(3y - x) = u$.

Now we are finally able to write out our integral!

$$\text{Total mass} = \iint_D \delta(x,y)\,dA = \iint_D \delta(u,v)\,|\mathrm{Jacobian}|\,dv\,du = \int_0^1\int_{-1}^0 u\,|-17|\,dv\,du = 17\int_0^1\int_{-1}^0 u\,dv\,du.$$

We compute: $= 17\int_0^1 uv\Big]_{-1}^0 du = 17\int_0^1 (0 - u(-1))\,du = 17\int_0^1 u\,du = \frac{17}{2}u^2\Big]_0^1 = \frac{17}{2} - 0 = \boxed{\frac{17}{2}\ \mathrm{kg}}$

3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.



When $Z=5$, we get $5 = 9 - x^2 - y^2$
$x^2+y^2 = 4$, which is a circle of radius 2.

The top plane is at the point where $x^2+y^2=1$, which implies the plane $Z=8$. Regardless, a conversion to polar coordinates seems appropriate.

$Z = 9 - r^2$ and when $Z=5$, we have $5 = 9-r^2$, so $r^2 = 4$, Therefore $r=2$, which is the bottom boundary of the area which we are trying to calculate.

Then we know that $x^2+y^2 = r^2$, so $r^2 = 1$, therefore $r=1$, which would be the upper boundary of the area because we are looking for the area outside of the cylinder and once $r<1$, we are inside the cylinder. And we are looking at a full circle, so θ is between 0 and 2π!

Our parametrization of the surface in polar coordinate is $G(r,\theta) = \langle r\cos\theta, r\sin\theta, 9-r^2 \rangle$
for $0 \le \theta \le 2\pi$, $1 \le r \le 2$.
$D: \{ 0 \le \theta \le 2\pi, 1 \le r \le 2 \}$.

Now, we need to compute our tangent and normal vectors to calculate the surface area!

$\frac{\partial g}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$ $\frac{\partial g}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$.

We compute the normal vector: $\vec{N} = \frac{\partial g}{\partial r} \times \frac{\partial g}{\partial \theta} = \det \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}$

$\vec{N} = (\sin\theta(0) - (-2r(r\cos\theta)))\vec{i} - (\cos\theta(0) - (-2r(-r\sin\theta)))\vec{j} + (r\cos^2\theta + r\sin^2\theta)\vec{k} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle$.

Surface area $= \iint_D \|\vec{N}(u,v)\| \, du \, dv$, so we need the magnitude of the normal vector!

$\|\vec{N}\| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = \sqrt{4r^4 + r^2} = \sqrt{r^2(4r^2+1)} = r\sqrt{4r^2+1}$.

Now we can compute the surface area. let $u = 4r^2+1$, $du = 8r$

$\int_0^{2\pi}\int_1^2 r\sqrt{4r^2+1}\, dr\, d\theta = \frac{1}{8}\int_0^{2\pi}\int_1^2 (4r^2+1)^{1/2}\, 8r\, dr\, d\theta = \frac{1}{8}\cdot\frac{2}{3}\int_0^{2\pi}\left[(4r^2+1)^{3/2}\right]_1^2 d\theta$

$= \frac{1}{12}\int_0^{2\pi}\left((17)^{3/2} - (5)^{3/2}\right)d\theta = \boxed{\frac{\pi}{6}\left(17^{3/2} - 5^{3/2}\right).}$ Therefore, the surface area is calculated!

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

(a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x,y)$, $g(x,y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_{\mathcal{C}} f\nabla g \cdot \mathbf{n}\, ds - \oint_{\mathcal{C}} g\nabla f \cdot \mathbf{n}\, ds.$$

(*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$) = Laplacian.

We know that the integration by parts formula is:

$$\iint_D f \operatorname{div} \vec{F}\, dA = \oint_C f\vec{F}\cdot\hat{n}\, ds - \iint_D \nabla f \cdot \vec{F}\, dA$$

We can apply this formula because we have a smooth, simple, closed, and oriented curve, which allows us to use Green's Theorem!

So we can rewrite the integrals as $\iint_D f\,\Delta g\, dA = \iint_D f \operatorname{div} \nabla g\, dA$ and $\iint_D g\Delta f\, dA = \iint_D g \operatorname{div} \nabla f\, dA$.

Starting with the first double integral, we can write it as the difference between a line integral and double integral by applying the integration by parts formula.

If we let $\vec{F} = \nabla g$, we get $\iint_D f\Delta g\, dA = \iint_D f \operatorname{div} \nabla g\, dA = \oint_C f\nabla g\cdot\hat{n}\, ds - \iint_D \nabla f\cdot\nabla g\, dA$.

Now we apply the integration by parts formula to the second double integral.

If we let $\vec{F} = \nabla f$, then we get $\iint_D g\Delta f\, dA = \oint_C g\nabla f\cdot\hat{n}\, ds - \iint_D \nabla g\cdot\nabla f\, dA$.

If we now take the difference between these two applications of the integration by parts formula, we should have adequately proven the above identity.

$$\iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \iint f \operatorname{div}\nabla g\, dA - \iint g \operatorname{div}\nabla f\, dA = \oint_C f\nabla g\cdot\hat{n}\, ds - \iint_D \nabla f\nabla g\, dA - \left[\oint_C g\nabla f\cdot\hat{n}\, ds - \iint_D \nabla g\nabla f\, dA\right]$$

Therefore, since, $\iint_D \nabla f\nabla g\, dA = \iint_D \nabla g\nabla f\, dA$, we can say that:

$$\boxed{\iint_D f\Delta g\, dA - \iint_D g\Delta f\, dA = \oint f\nabla g\cdot\hat{n}\, ds - \oint g\nabla f\cdot\hat{n}\, ds}$$

Which is the identity that we were attempting to prove. Q.E.D.

(b). Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu f \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in C,$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in C.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

From part a): $\iint_D f\Delta g\,dA - \iint_D g\Delta f\,dA = \oint_C f\nabla g\cdot\hat{n}\,ds - \oint_C g\nabla f\cdot\hat{n}\,ds$

if $\Delta f = \lambda f$ and $\Delta g = \mu f$, for $(x,y) \in D$ we can write the above statement as:

~~$\iint_D f(\mu f)\,dA$~~ $- \iint_D g\lambda f\,dA$ because it is on the domain D.

$\mu\iint_D f^2\,dA - \lambda\iint_D g f\,dA = \oint_C f\nabla g\cdot\hat{n}\,ds - \oint_C g\nabla f\cdot\hat{n}\,ds$

0 because $f=0$ for $(x,y) \in$; 0 because $g=0$ for $(x,y) \in$.

Now, we have

$\mu\iint_D f^2\,dA - \lambda\iint_D f(x,y)g(x,y)\,dA = 0$, so we almost have the solution that we need.

Now, we need to prove that $\mu\iint_D f^2\,dA = 0$. We know $\iint_D f\Delta g\,dA = \oint f\nabla g\cdot\hat{n}\,ds - \iint_D \nabla f\cdot\nabla g\,dA$ (0 for $(x,y) \in C$.)

So $\mu\iint_D f^2\,dA = -\iint_D \nabla f\cdot\nabla g\,dA$

or $\iint_D f\,\text{div}\,\nabla g = -\iint_D \nabla f\cdot\nabla g\,dA = 0$ on the domain D because $\mu \leq 0$ and f^2 is always positive on the domain D.

Therefore $-\lambda\iint_D f(x,y)g(x,y)\,dA = 0$

So $\iint_D f(x,y)g(x,y)\,dA = 0$