Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
2/22/2019

Name: ____________________
SID: ____________________
TA Section: ____________________

Time Limit: 50 Minutes

Version ($\uparrow$)

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\, dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\, dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\, p_X(x)\, dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x, y)$ then
  - The total probability $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p_{X,Y}(x, y)\, dxdy = 1$
  - The probability that $(X, Y) \in \mathcal{D}$ is $\mathbb{P}[(X, Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x, y)\, dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X, Y)$ is $\mathbb{E}[f(X, Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, p_{X,Y}(x, y)\, dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \leq x \leq 2\ ,\ -1 \leq y \leq 1\}$ oriented counterclockwise and let

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y\, ,\, \sin(e^y) + 3x^2y \right\rangle .$$

Evaluate the line integral

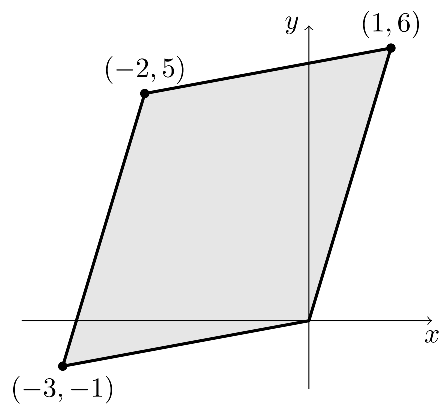
$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Solution: Applying Green's Theorem followed by Fubini's Theorem we have

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{D}} \operatorname{curl}_z \mathbf{F}\, dA = \int_{-1}^{1} \int_{-2}^{2} (6xy + 4)\ dxdy = \int_{-1}^{1} 16\, dy = 32.$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1),(0,0),(1,6),(-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y)=\frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.

Solution: We first draw a picture of the domain:



We observe that the vector from $(-3,-1)$ to $(0,0)$ is $\langle 3,1\rangle$ and the vector from $(-3,-1)$ to $(-2,5)$ is $\langle 1,6\rangle$. This motivates the change of variables

$$x = 3u+v,$$
$$y = u+6v,$$

which may be inverted to obtain

$$u = \frac{1}{17}\left(6x-y\right),$$
$$v = \frac{1}{17}\left(3y-x\right).$$

In the (u,v) coordinates we then have

$$\mathcal{D}=\{-1\le u\le 0\ ,\ 0\le v\le 1\}\,.$$

Next we compute the Jacobian for this change of variables

$$\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{bmatrix}3 & 1\\ 1 & 6\end{bmatrix}=17.$$

We also observe that the density function

$$\delta(x,y)=\frac{1}{17}\left(3y-x\right)=v.$$

As a consequence, we have

$$\text{Total Mass} = \iint_{\mathcal{D}}\delta(x,y)\,dA=\int_{-1}^{0}\int_{0}^{1}17v\,dvdu=\frac{17}{2}\ \text{kg}$$

3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.

Solution: Denote the surface by $\mathcal{S}$. We note that the paraboloid intersects the plane $z = 5$ when $x^2+y^2 = 4$ and hence $\mathcal{S}$ has projection in the (x, y) plane given by $\{1 \leq x^2+y^2 \leq 4\}$.

Next we parameterize $\mathcal{S}$ using

$$G(r,\theta) = \left(r\cos\theta, r\sin\theta, 9 - r^2\right),$$

for $0 \leq \theta < 2\pi$ and $1 \leq r \leq 2$.

We then compute

$$\frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle,$$
$$\frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle,$$

and so

$$\frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} = \left\langle 2r^2\cos\theta, 2r^2\sin\theta, r \right\rangle,$$

which has magnitude

$$\left\| \frac{\partial G}{\partial r} \times \frac{\partial G}{\partial \theta} \right\| = r\sqrt{4r^2+1}.$$

As a consequence,

$$\text{Area}(\mathcal{S}) = \iint_{\mathcal{S}} dS = \int_0^{2\pi} \int_1^2 r\sqrt{4r^2+1}\, dr d\theta = \frac{\pi}{6}\left(17^{\frac{3}{2}} - 5^{\frac{3}{2}}\right).$$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

   (a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x,y)$, $g(x,y)$ we have the identity

$$\iint_{\mathcal{D}} f\Delta g\,dA - \iint_{\mathcal{D}} g\Delta f\,dA = \oint_{\mathcal{C}} f\nabla g\cdot\mathbf{n}\,ds - \oint_{\mathcal{C}} g\nabla f\cdot\mathbf{n}\,ds.$$

   (*Hint: Recall that* $\Delta f = \operatorname{div}\nabla f$)

Solution: Applying the integration by parts formula to the first term on the left hand side we obtain

$$\iint_{\mathcal{D}} f\Delta g\,dA = \oint_{\mathcal{C}} f\nabla g\cdot\mathbf{n}\,ds - \iint_{\mathcal{D}} \nabla f\cdot\nabla g\,dA.$$

Applying the integration by parts formula to the second term on the left hand side we obtain

$$\iint_{\mathcal{D}} g\Delta f\,dA = \oint_{\mathcal{C}} g\nabla f\cdot\mathbf{n}\,ds - \iint_{\mathcal{D}} \nabla f\cdot\nabla g\,dA.$$

Taking the difference of these two expressions we obtain the desired identity.

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

$$\begin{aligned} \Delta f &= \lambda f \quad \text{for all } (x,y) \in \mathcal{D}, \\ \Delta g &= \mu g \quad \text{for all } (x,y) \in \mathcal{D}, \end{aligned}$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$\begin{aligned} f(x,y) &= 0 \quad \text{for all } (x,y) \in \mathcal{C}, \\ g(x,y) &= 0 \quad \text{for all } (x,y) \in \mathcal{C}. \end{aligned}$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

Solution: Using the boundary conditions and part (a) we have

$$\iint_{\mathcal{D}} f\Delta g\,dA - \iint_{\mathcal{D}} g\Delta f\,dA = 0.$$

Using the equations satisfied by f, g we then obtain

$$(\lambda - \mu)\iint_{\mathcal{D}} fg\,dA = \iint_{\mathcal{D}} f\Delta g\,dA - \iint_{\mathcal{D}} g\Delta f\,dA = 0.$$

As $\lambda \neq \mu$ we must then have that

$$\iint_{\mathcal{D}} fg\,dA = 0,$$

as required.