Math 32B - Lectures 3 & 4
Winter 2019
Midterm 2
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TA Section: 4B

Time Limit: 50 Minutes

Version (↑)

This exam contains 12 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

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Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$

1. (6 points) Let $\mathcal{C}$ be the boundary of the rectangle $\mathcal{D} = \{-2 \le x \le 2\,,\ -1 \le y \le 1\}$ oriented counterclockwise and let

right direction for Green's theorem

$$\mathbf{F}(x,y) = \left\langle e^{x^3-x} - 4y,\ \sin(e^y) + 3x^2y \right\rangle.$$

Evaluate the line integral

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

– use green's theorem with curl$_2$

By def, $\iint_D \text{curl}_2 F = \oint_C F \cdot dr$

$$\int_{-1}^{1}\int_{-2}^{2} \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dx\, dy$$

we compute

$$\int_{-1}^{1}\int_{-2}^{2} 6xy + 4 \quad dx\, dy$$

$$3x^2y + 4x \Big|_{-2}^{2}$$

$$3(4)y + 8$$

$$-\ 3(4)y + 8$$

$$\int_{-1}^{1} 16\ dy$$

$$16(1+1) = \boxed{32}$$

$$\int_{-1}^{1}\int_{-2}^{2} 6xy + 4 \, dx\, dy$$

$$3x^2y + 4x \Big|_{-2}^{2}$$

$$12y + 8$$

$$-12y + 8$$

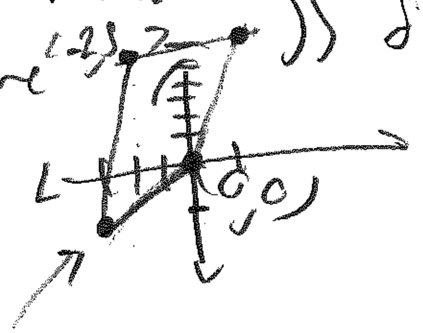
$$\int_{-1} 16 \, dy \quad \boxed{32}$$

2. (14 points) The lamina $\mathcal{D}$ is a parallelogram with corners $(-3,-1), (0,0), (1,6), (-2,5)$ (where distance is measured in meters) and with mass density $\delta(x,y) = \frac{1}{17}(3y-x)\ \mathrm{kg\,m^{-2}}$. Find the total mass of $\mathcal{D}$.

change of variables → jacobian use formula

By def mass = $\iint \delta(x,y)$

Draw a picture



$17u = 3y - x$

$0, 0$

$\langle -3, -1 \rangle$

compute the jacobian

Apply linear system: $u\langle 1,6\rangle + v\langle 3,1\rangle$

$x = u + 3v \qquad y = 6u + v$

$\frac{\partial(x,y)}{\partial(u,v)}$

Find Jacobian $\begin{vmatrix} 1 & 3 \\ 6 & 1 \end{vmatrix} = 1 - 18 = |-17| = 17 \quad \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix}$

Jacobian $= 17$

$dx\,dy = 17\,du\,dv$

We compute $\iint 3y - x \; du\,dv$

$3(6u+v) - u - 3v$

$18u + 3v - u - 3v = 17u$

which becomes $\int_0^1\int_0^1 17u \,\frac{17}{17}\, du\,dv$

$(0,1)$ as one word ~ more length 1 and shifted to $(0,0)$

$\frac{17u^2}{2}\Big|_0^1$

$\int_0^1 \frac{17}{2}\,dv = \boxed{\frac{17}{2} \text{ units}}$

$17u = 3y - x$ $u = \frac{3y - x}{17}$ $-6(x = u + 3v)$ $-18u$

$y = 6u + v$ $+v$

$\frac{y - 6x}{17} = \frac{-17v}{-17}$

$u = \frac{3y - x}{17}$ $v = \frac{-y + 6x}{17}$

plugin points

$(-3, -1)$ $(0, 0)$ $(1, 0)$

x y

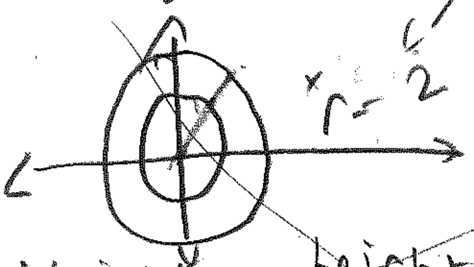
$-3 + 3$

$u = 0$

3. (12 points) Find the area of the part of the paraboloid $z = 9 - x^2 - y^2$ outside the cylinder $x^2 + y^2 = 1$ and above the plane $z = 5$.

Draw a picture

keep paper picture over 1 side



$5 = 9 - x^2 - y^2$

$-4 \quad -9$

$x^2 + y^2 = 4$

Find height

~~height = $9 - x^2 - y^2 - 5$~~

height = $4 - x^2 - y^2$ to polar coordinates

$$\int_0^{2\pi}\int_1^2 r(4 - r^2)\, dr\, d\theta$$

$$\int_0^{2\pi}\int_1^2 4r - r^3\, dr\, d\theta$$

$$2r^2 - \frac{r^4}{4}\Big|_1^2$$

$$8 - 4 - 2 + \frac{1}{4}$$

$$2\frac{1}{4}$$

We compute

$$\int_0^{2\pi} \frac{9}{4}\, d\theta$$

$$\boxed{\frac{9\pi}{2}\, u^2}$$

Draw a picture → another sheet

$z = 9 - x^2 - y^2$ $x^2 + y^2 = 1$

↗ parameterize

$G(x, y) = \langle x, y, 9 - x^2 - y^2 \rangle$ $4r^2$

$\frac{dG}{dx} = \langle 1, 0, -2x \rangle$

$\frac{dG}{dy} = \langle 0, 1, -2y \rangle$

scalar field where value = 1

$\langle 2x, +2y, 1 \rangle$

formula for area of surface

$\iint_D 1 \left\| \frac{dG}{dx} \times \frac{dG}{dy} \right\| dx\, dy$ ↓ to polar coordinates

$\iint_D \sqrt{4x^2 + 4y^2 + 1}\, dx\, dy$

$\sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\; r\, dr\, d\theta$

use u substitution

$\int_0^{2\pi} \int_0^2 r\sqrt{4r^2+1}\, dr\, d\theta$

$u = 4r^2 + 1$

$\frac{du}{8} = \frac{8r}{8}$

$\int_0^{2\pi} \int_1 \frac{1}{8} u^{1/2}$

$\frac{2}{3} \cdot \frac{1}{8}$

$\int_0^{2\pi} \frac{1}{12} (4r^2+1)^{3/2} \Big|^2$

$\frac{1}{12} (17^{3/2} - 5^{3/2}) = \boxed{\frac{\pi}{6} (17^{3/2} - 5^{3/2})}$

4. (8 points) Let $\mathcal{D} \subset \mathbb{R}^2$ be bounded by a smooth, simple, closed curve $\mathcal{C}$ oriented counterclockwise, with outward pointing unit normal $\mathbf{n}$.

(a) Using the integration by parts formula or otherwise, show that for smooth scalar functions $f(x, y)$, $g(x, y)$ we have the identity

div

$$\iint_{\mathcal{D}} f\Delta g\, dA - \iint_{\mathcal{D}} g\Delta f\, dA = \oint_{\mathcal{C}} f\nabla g \cdot \mathbf{n}\, ds - \oint_{\mathcal{C}} g\nabla f \cdot \mathbf{n}\, ds.$$

(*Hint: Recall that* $\Delta f = \operatorname{div} \nabla f$)

$= \oint_C f \operatorname{div} \nabla g_{n} - g \operatorname{div} \nabla f_{n}$

Integration by parts formula

set $F = \nabla g$, $\operatorname{div} \nabla g$

$\iint_D f \operatorname{div} F = \oint_C f\, F \cdot \hat{n}\, ds - \iint_D \nabla f \cdot F\, dA$

$\iint f \operatorname{div} \nabla g - g \operatorname{div} \nabla f = \oint_C f \nabla g \cdot n\, ds - \oint_C g \nabla f \cdot n$

plug in

$\iint_D f \operatorname{div} \nabla g + \iint_D \nabla f \cdot \nabla g\, dA = \oint_C f \nabla g \cdot \hat{n}\, ds$

✓ by parts again

isolate

split apart

$\iint_D f \operatorname{div} \nabla g - \iint g \operatorname{div} \nabla f + \oint_C g \nabla f \cdot n\, ds = \oint_C f \nabla g \cdot n$

therefore $\therefore \iint_D f \Delta g - \iint_D g \Delta f\, dA = \oint_C f \nabla g \cdot n\, ds - \oint_C \nabla f \cdot n\, ds$

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle$$ (div)

$$-g\left(\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right) + g \nabla f \cdot n$$

$$g \left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle \cdot \left\langle -\frac{df}{dy}, \frac{df}{dx} \right\rangle$$

$$-g\left(\frac{d^2f}{dx^2} + \frac{d^2f}{dy^2}\right) + g\left(\frac{-d^2f}{dxdy} + \frac{d^2f}{dxdy}\right)$$

$$\left\langle \frac{df}{dx}, \frac{df}{dy} \right\rangle \cdot \left\langle \frac{dg}{dx}, \frac{dg}{dy} \right\rangle$$

$$\iint_D f \operatorname{div} \nabla g - g \operatorname{div} \nabla f = \oint_C f \nabla g \cdot n \, ds$$

$$\oint_C f \nabla g \cdot \left\langle -\frac{dg}{dy}, \frac{dg}{dx} \right\rangle$$

$$\iint_D (f \operatorname{div} \nabla g - g \operatorname{div} \nabla f) = 0$$

$$+ \oint_C g \nabla f \cdot \left\langle -\frac{df}{dy}, \frac{df}{dx} \right\rangle$$

$$\iint_D f \operatorname{div} \nabla f = \iint g \operatorname{div} \nabla f$$

$$\oint_C g\left(\frac{-d^2f}{dxdx} + \frac{d^2f}{dxdy}\right)$$

0 by symmetry over curve is also 0

substitute in with integration by parts

(b) Suppose that $f(x,y)$, $g(x,y)$ are smooth, non-zero, scalar functions satisfying the equations

div $\nabla f = \lambda$

$$\Delta f = \lambda f \quad \text{for all } (x,y) \in \mathcal{D},$$
$$\Delta g = \mu g \quad \text{for all } (x,y) \in \mathcal{D},$$

where $\lambda, \mu \leq 0$ are real numbers. Suppose also that $f(x,y)$, $g(x,y)$ satisfy the boundary condition

$$f(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C},$$
$$g(x,y) = 0 \quad \text{for all } (x,y) \in \mathcal{C}.$$

Using your answer to part (a), show that whenever $\lambda \neq \mu$ we have

$$\iint_{\mathcal{D}} f(x,y)g(x,y)\,dA = 0.$$

f = 0 along curve

g = 0 along curve

Given $\iint_D f \Delta g \, dA - \iint_D g \Delta f \, dA = \oint_C f \nabla g \cdot n \, ds - \oint_C g \nabla f \cdot n \, ds$

$\iint_D f \mu g - \iint_D g \lambda f = 0$

can combine same domain

$\iint f \mu g - g \lambda f \, dA$ constant so can divide out

$\iint_D fg(\mu - \lambda) \, dA = 0$

Therefore $\iint_D fg \, dA = 0$