Math 32B — Lectures 1 & 2
Spring 2022
Midterm 2
5/18/2022

Name: ______________________________

UID: ______________________________

Time Limit: 50 mins

Version $\rightarrow$

This exam contains 9 pages (including this cover page) and 3 problems. There are a total of 25 points available.

Check to see if any pages are missing. Enter your name and UID at the top of this page.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions legibly, in full English sentences, using units where appropriate.
- **For full credit, be sure to clearly explain your answer, showing all working.**
- **All answers should be written in this answer booklet.** You may write on both sides of each page if required.
- You are allowed one handwritten side of US letter paper with notes. You may **not** use any other notes, books, calculators, or other assistance. Please do not turn in your note sheet.
- You may use scratch paper if required. Please do not turn in your scratch paper.

1. (7 points) Let $\mathscr{C}$ be the curve with parameterization

$$\mathbf{r}(t) = \big\langle 2t - 2\sin(t)\,,\, 2 - 2\cos(t)\big\rangle \quad \text{for} \quad 0 \le t \le 2\pi.$$

   (a) Find the length of $\mathscr{C}$.

   (b) Find $\displaystyle\int_{\mathscr{C}} \langle y, 0\rangle \cdot d\mathbf{r}$.

<u>*Hint:*</u> *You may wish to use the identities*

$$\sin^2(u) = \tfrac{1}{2}\big[1 - \cos(2u)\big] \quad \textit{and} \quad \cos^2(u) = \tfrac{1}{2}\big[1 + \cos(2u)\big].$$

Solution:

(a) We compute that

$$\mathbf{r}'(t) = \big\langle 2 - 2\cos(t)\,,\, 2\sin(t)\big\rangle,$$

and hence (using the hint)

$$\|\mathbf{r}'\| = \sqrt{8 - 8\cos(t)} = 4\sin\big(\tfrac{t}{2}\big).$$

We then have

$$\text{Length}(\mathscr{C}) = \int_0^{2\pi} 4\sin\big(\tfrac{t}{2}\big)\,dt = 16.$$

(b) Similarly (and again using the hint):

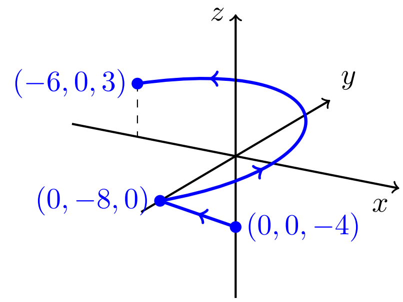
$$\begin{aligned}\int_{\mathscr{C}} \langle y, 0\rangle \cdot d\mathbf{r} &= \int_0^{2\pi} \big[2 - 2\cos(t)\big]^2\,dt \\ &= \int_0^{2\pi} 4 - 8\cos(t) + 4\cos^2(t)\,dt \\ &= \int_0^{2\pi} 6 - 8\cos(t) + 2\cos(2t)\,dt \\ &= \Big[6t - 8\sin(t) + \sin(2t)\Big]_{t=0}^{t=2\pi} \\ &= 12\pi.\end{aligned}$$

2. (8 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}} + \cos(y)\,,\ -(x+z)\sin(y)\,,\ \cos(y) + e^z \right\rangle.$$

(a) Show that $\mathbf{F}$ is a conservative vector field by finding a potential function for $\mathbf{F}$.

(b) Let $\mathscr{C}$ be the curve shown in the following sketch:



Find $\displaystyle\int_{\mathscr{C}} \mathbf{F}\cdot d\mathbf{r}$.

Solution:

(a) We will find a potential function f so that $\mathbf{F} = \nabla f$.

Comparing the x-components, we see that

$$\frac{x}{\sqrt{1+x^2}} + \cos(y) = \frac{\partial f}{\partial x},$$

and integrating we must have that

$$f = \sqrt{1+x^2} + x\cos(y) + A(y,z),$$

for some function $A(y,z)$.

Comparing the y-components, we see that

$$-(x+z)\sin(y) = \frac{\partial f}{\partial y} = -x\sin y + \frac{\partial A}{\partial y},$$

and hence

$$A(y,z) = z\cos(y) + B(z),$$

for some function $B(z)$.

Finally, comparing the z-components we see that

$$\cos(y) + e^z = \frac{\partial f}{\partial z} = \cos(y) + \frac{dB}{dz},$$

and hence

$$B(z) = e^z$$

will suffice.

In conclusion,

$$f(x,y,z) = \sqrt{1+x^2} + (x+z)\cos(y) + e^z$$

is a potential function for $\mathbf{F}(x,y,z)$ and hence $\mathbf{F}$ is conservative.

(b) Taking $f(x, y, z)$ is in part (a), we apply the Fundamental Theorem for Conservative Vector Fields to get

$$\int_{\mathscr{C}} \mathbf{F} \cdot d\mathbf{r} = f(-6, 0, 3) - f(0, 0, -4) = \left(\sqrt{37} - 3 + e^3\right) - \left(1 - 4 + e^{-4}\right) = \sqrt{37} + e^3 - e^{-4}.$$

3. (10 points) Let $\mathcal{S}$ be the part of the sphere $x^2+y^2+z^2=4$ where $-x \le y \le x$ and $z \ge 1$, oriented with the downwards pointing normal. Find the flux of

$$\mathbf{F}(x,y,z) = \left\langle \frac{xz}{x^2+y^2+z^2}\,,\ \frac{yz}{x^2+y^2+z^2}\,,\ 0 \right\rangle$$

across $\mathcal{S}$.

Solution: We parameterize $\mathcal{S}$ using

$$G(\theta,\phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi\rangle\,,$$

for $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$ and $0 \le \phi \le \frac{\pi}{3}$. We compute that the downwards normal is

$$\begin{aligned}\mathbf{N} &= \frac{\partial G}{\partial\theta} \times \frac{\partial G}{\partial\phi}\\ &= \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0\rangle \times \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi\rangle\\ &= -4\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\,.\end{aligned}$$

We then have

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\frac{\pi}{3}} \langle\cos\theta\sin\phi\cos\phi, \sin\theta\sin\phi\cos\phi, 0\rangle \cdot \Big[-4\sin\phi\,\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle\Big]\,d\phi\,d\theta\\ &= -\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{\frac{\pi}{3}} 4\sin^3\phi\cos\phi\,d\phi\,d\theta\\ &= -\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left[\sin^4\phi\right]_{\phi=0}^{\phi=\frac{\pi}{3}}\,d\theta\\ &= -\tfrac{9\pi}{32}.\end{aligned}$$