MATH 32B Midterm 2Y

Rishab Sukumar

TOTAL POINTS

45 / 50

QUESTION 1

1 Line integral 8 / 8

✓ + 1 pts **A correct parametrization of circle (bounds for parameter must be given at start, but do not need to be correct [though they need to make sense]).**

✓ + 2 pts **Correct bounds on parameter given either with parametrization or in bounds of integral (separate from points for orientation; does not count if incorrect and yields correct answer)**

✓ + 1 pts **Correct counter-clockwise orientation of curve (as determined by bounds on parameter given either immediately after parametrization or in bounds of integral)**

✓ + 2 pts **Attempted vector line integral**

✓ + 1 pts **Correct substitution of component functions into vector field and computation of integral**

✓ + 1 pts **Readable & fully (correctly or incorrectly) explained. Point not awarded if same letter used twice to mean different things**

+ **0 pts** No points (e.g. if attempted surface integral or surface parametrization; points awarded for curve parametrization even if attempted scalar line integral)

+ **1.5 pts** Chose clockwise orientation but correctly explained need to change sign for final answer (does not count if faulty explanation yields correct answer)

QUESTION 2

2 Fundamental theorem of vector line integrals 9 / 10

✓ + 7 pts **Correct potential function f(x,y,z) = -x*sin(y*z) + y^2 + z**

✓ + 2 pts **Integral is equal to f(1,pi,1) - f(0,0,1)**

+ **1 pts** Correct final answer of pi^2, correctly derived using potential function

+ **0 pts** No points

+ **1 pts** (Incorrect) statement that integral is equal to f(0,0,1) - f(1,pi,1)

+ **2 pts** Expression for a parametric curve from (0,0,1) to (1,pi,1) (only if no solution via potential function)

+ **4 pts** Correct expression for vector line integral in terms of a parametric curve (only if no solution via potential function)

+ **4 pts** Correct answer of pi^2, correctly derived by doing a vector line integral using a parametric curve

+ **2 pts** Partial credit for an incomplete or mildly incorrect expression for a vector line integral using a parametric curve (only if no solution via potential function)

+ **1 pts** Partial credit for progress towards doing the integral in a vector line integral using a parametric curve (only if no solution via potential function)

Note: starting point is (0,0,1), not (0,0,0)

QUESTION 3

3 Change of variables 15 / 16

✓ + 4 pts **Correct computation of either $$\frac{\partial(u,v)}{\partial(x,y)}$$ or $$\frac{\partial(x,y)}{\partial(u,v)}$$.**

✓ + 4 pts **Correctly converting the integrand into $$u,v$$ coordinates.**

✓ + 4 pts **Correctly replacing $$dxdy$$ by $$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$.**

✓ + 2 pts **Correct limits for $$u,v$$.**

+ **1 pts** Correct answer of $$6\ln2\,\mathrm{kg}\,\mathrm{m}^2$$ (including units).

✓ + 1 pts **Answer clearly explained, using full english sentences.**

+ **3 pts** Matrix defining Jacobian correct but determiniant incorrectly computed. (Partial credit)

+ **2 pts** Matrix defining Jacobian partly correct. (Partial credit)

+ **2 pts** Integrand partly converted into $$u,v$$ coordinates correctly. (Partial credit)

+ **2 pts** Replaced $$dxdy$$ by $$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|dudv$$ instead of $$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$. Requires Jacobian to be correct for credit. (Partial credit)

+ **2 pts** Attempted to correctly replace $$dxdy$$ by $$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|dudv$$ but made error in converting $$\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$$ to $$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$$. Requires Jacobian to be correct for credit. (Partial credit)

+ **0 pts** No credit due.

Units missing.

QUESTION 4

4 Surface integral 13 / 16

✓ + **1 pts Readability**

✓ + **2 pts Valid Parametrization**

✓ + **2 pts Parameter Domain**

✓ + **2 pts First Tangent Vector**

✓ + **2 pts Second Tangent Vector**

+ **2 pts** Normal Vector

✓ + **2 pts Length of Normal Vector**

✓ + **2 pts Integral**

+ **1 pts** Correct Answer

+ **0 pts** Placeholder

Normal Vector: Dropped third component.

Math 32B - Lecture 1
Fall 2018
Midterm 2
11/14/2018

Name: RISHAB SUKUMAR
TA Section: 1B
UID: 304902259

Time Limit: 50 Minutes

Version Y

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 50 points available.

Check to see if any pages are missing. Enter your name and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

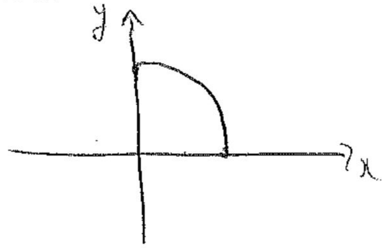
Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$
- If $\mathcal{W}$ is a solid with mass density $\delta(x,y,z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x,y,z)\,dV$
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\,\delta(x,y,z)\,dV$
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\,\delta(x,y,z)\,dV$
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\,\delta(x,y,z)\,dV$
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2+z^2)\,\delta(x,y,z)\,dV$
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2+z^2)\,\delta(x,y,z)\,dV$
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2+y^2)\,\delta(x,y,z)\,dV$

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$
  - The marginal probability density function of X is $p_X(x) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dy$
  - The marginal probability density function of Y is $p_Y(y) = \int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dx$

1. (8 points) Find $\int_C y\,dx - x\,dy$ where C is the part of the unit circle in the first quadrant, oriented **counterclockwise.**



Parametric function:

$r(t) = \langle \cos t, \sin t \rangle$

$0 \le t \le \frac{\pi}{2}$

$r'(t) = \langle -\sin t, \cos t \rangle$

$\|r'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1$

$\therefore dx = -\sin t\, dt$

$dy = \cos t\, dt$

$\int_C y\,dx - x\,dy$

Substitute values for x and y with parameterization

$= \int_0^{\frac{\pi}{2}} \sin t \cdot (-\sin t)\,dt - \cos t\,(\cos t)\,dt$

$= \int_0^{\frac{\pi}{2}} -\sin^2 t - \cos^2 t \; dt$

$= -\int_0^{\frac{\pi}{2}} \sin^2 t + \cos^2 t \; dt$

$= -\int_0^{\frac{\pi}{2}} 1 \; dt$

$= -t \Big|_0^{\pi/2}$

$= -\frac{\pi}{2}$

2. (10 points) Let $\mathbf{F}(x, y, z) = \langle -\sin(yz), -xz\cos(yz) + 2y, -xy\cos(yz) + 1\rangle$ and $\mathcal{C}$ be a smooth curve from $(0, 0, 1)$ to $(1, \pi, 1)$. Find $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Curve from $(0,0,1)$ to $(1,\pi,1)$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\sin(yz) & -xz(\cos(yz)) + 2y & -xy\cos(yz)+1 \end{vmatrix}$$

$$= i\left(\frac{\partial}{\partial y}\left[-xy\cos(yz)+1\right] - \frac{\partial}{\partial z}\left(-xz\cos(yz)+2y\right)\right)$$
$$-j\left(\frac{\partial}{\partial x}\left(-xy\cos(yz)+1\right) - \frac{\partial}{\partial z}\left(-\sin(yz)\right)\right)$$
$$+k\left(\frac{\partial}{\partial x}\left(-xz(\cos(yz))+2y\right) - \frac{\partial}{\partial y}\left(-\sin(yz)\right)\right)$$

$$= i\left(-x\cos yz + xyz\sin(yz) - \left(-x\cos(yz) + xyz\sin(yz)\right)\right)$$
$$-j\left(\cancel{-y\cos(yz)} + \cancel{y\cos(yz)}\right)$$
$$+k\left(\cancel{-z\cos(yz)} + \cancel{z\cos(yz)}\right)$$

$$= 0i - 0j + 0k$$
$$= 0$$

Thus, F is a conservative vector field

Since F is a conservative field, $F = \nabla f$

$$\frac{\partial f}{\partial x} = -\sin yz$$

$$\partial f = -\sin yz \,\partial x$$

$$\int \partial f = \int -\sin(y\cancel{z})\ dx$$

$$f = -x\sin(yz) + h(y) + g(z)$$

$$\frac{\partial f}{\partial y} = -xz\cos(yz) + 2y$$

$$\partial f = -xz\cos(yz) + 2y\ \partial y$$

$$\int \partial f = \int -xz\cos(yz) + 2y\ dy$$

$$f = \frac{-x\cancel{z}\sin(yz)}{\cancel{z}} + y^2 + g(z) + i(x)$$

$$= -x\sin(yz) + y^2 + g(z) + i(x)$$

$$\frac{\partial f}{\partial z} = -xy\cos(yz) + 1$$

$$\partial f = -xy\cos(yz) + 1\ \partial z$$

$$\int \partial f = \int -xy\cos(yz) + 1\ \partial z$$

$$f = -x\sin(yz) + z + h(y) + i(x)$$

$$-x\sin(yz) + y^2 + g(z) + i(x) = -x\sin(yz) + z + h(y) + ix$$

For a conservative field,

$$\int_C F\cdot dr = f(Q) - f(P)$$

$$\therefore g(z) = z$$

$$f = -x\sin(yz) + y^2 + z$$

$$\int_C F\cdot dr = f(1,\pi,1) - f(0,0,0)$$

$$= -1\sin(\pi) + \pi^2 + 1 - (0+0+0)$$

$$= -1(0) + \pi^2 + 1 - 0 = \pi^2 + 1$$

3. (16 points) Let $\mathcal{D}$ be a lamina contained in the region $\left\{2 \leq xy \leq 4,\ 2 \leq \frac{y}{x} \leq 4,\ x \geq 0,\ y \geq 0\right\}$ (where distance is measure in metres) with density $\delta(x, y) = \dfrac{2y}{x}$ kilograms per square metre. Find the moment of inertia of $\mathcal{D}$ about the y-axis.

$\left(\textit{Hint: You might wish to use the variables } u = xy \textit{ and } v = \frac{y}{x}.\right)$

let $u = xy \qquad V = \frac{y}{x}$

$\therefore 2 \leq u \leq 4 \quad , 2 \leq v \leq 4$

$I_y = \iint_D x^2 \delta(x,y)\, dA = \iint_D = x^2 \delta(x,y)\, dx\, dy$

$F(x,y) = \left(xy, \frac{y}{x} \right)$

$$\text{Jac}(F) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix}$$

$$= \frac{y}{x} + \frac{y}{x}$$

$$= \frac{2y}{x}$$

By change of variables $\quad \text{Jac}(G) = \frac{x}{2y}$

$I_y = \iint_D x^2 \delta(x,y)\, dx\, dy = \iint_R x^2 \cdot \delta(x,y) \, \|\text{Jac}(G)\|\, du\, dv$

$= \iint_R x^2 \cdot \cancel{\frac{2y}{x}} \cdot \cancel{\frac{x}{2y}}\, du\, dv$

$$= \iint_R x^2 \, du\, dv$$

Note that $\frac{u}{v} = \frac{xy}{y} \cdot x = x^2$

$$= \iint_R \frac{u}{v} \, du\, dv$$

Limits for u and v were given in the question

$$= \int_2^4 \int_2^4 \frac{u}{v} \, du\, dv$$

$$= \int_2^4 \frac{u^2}{2v} \Big|_2^4 \, dv$$

$$= \int_2^4 \frac{\cancel{16}^{8}}{\cancel{2}v} - \frac{\cancel{4}^{2}}{\cancel{2}v} \, dv$$

$$= \int_2^4 \frac{8}{v} - \frac{2}{v} \, dv$$

$$= \int_2^4 \frac{6}{v} \, dv$$

$$= 6 \ln|v| \Big|_2^4$$

$$= 6[\ln|4| - \ln|2|]$$

$$= 6 \ln\left|\frac{4}{2}\right|$$

$$I_y = 6 \ln|2|$$

4. (16 points) Let the surface S be the part of the cone $z = 1 - \sqrt{x^2+y^2}$ where $z \geq 0$ and let $f(x,y,z) = z$. Find $\iint_S f(x,y,z)\, dS$.

$$z = 1 - \sqrt{x^2+y^2}$$

$$g(x,y) = z = 1 - \sqrt{x^2+y^2}$$

$$G(x,y) = (x, y, 1 - \sqrt{x^2+y^2})$$

$$T_x = \left\langle 1, 0, \frac{-\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} \right\rangle$$

$$= \left\langle 1, 0, \frac{-x}{\sqrt{x^2+y^2}} \right\rangle$$

$$T_y = \left\langle 0, 1, \frac{-\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2}} \right\rangle$$

$$= \left\langle 0, 1, \frac{-y}{\sqrt{x^2+y^2}} \right\rangle$$

$$N = T_x \times T_y$$

$$= \begin{vmatrix} i & j & k \\ 1 & 0 & \frac{-x}{\sqrt{x^2+y^2}} \\ 0 & 1 & \frac{-y}{\sqrt{x^2+y^2}} \end{vmatrix}$$

$$= i\left(0 + \frac{x}{\sqrt{x^2+y^2}}\right) - j\left(\frac{-y}{\sqrt{x^2+y^2}}\right) + k(1)$$

$$= \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$$

$$\|N\| = \sqrt{\frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2}} = \sqrt{\frac{x^2+y^2}{x^2+y^2}} = 1$$

$$\iint_S f(x,y,z)\, dS$$

$$= \iint_S f(x,y,z) \cdot \|N(x,y)\|\, dx\, dy$$

$$= \iint_S z \cdot 1 \; dx\, dy$$

$$= \iint_S 1 - \sqrt{x^2+y^2} \; dx\, dy$$

let $x^2+y^2 = r^2$

$\therefore z = 1 - \sqrt{r^2} = 1 - r$

Since, $z \geq 0 \quad \therefore 0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

by change of variables

$$= \int_0^{2\pi} \int_0^1 (1 - r)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 r - r^2 \; dr\, d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^3}{3} \Big|_0^1 \; d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} - \frac{1}{3} - 0 + 0 \; d\theta$$

$$= \int_0^{2\pi} \frac{3-2}{6} \; d\theta$$

$$= \int_0^{2\pi} \frac{1}{6} \; d\theta$$

$$\iint_S f(x,y,z)\, dS = \frac{1}{6}\theta \Big|_0^{2\pi} = \frac{2\pi}{6} = \frac{\pi}{3}$$