21W-MATH32B-4 MIDTERM 1

MATTHEW GUAN

TOTAL POINTS

30 / 30

QUESTION 1

1 Honor statement 0 / 0

✓ + 0 pts Complete

QUESTION 2

2 Question 1 4 / 4

✓ + 1 pts **Correct bounds for $$x$$**

✓ + 1 pts **Correct bounds for $$y$$**

✓ + 1 pts **Correct bounds for $$z$$**

✓ + 1 pts **Correct answer of $$\frac{15}8$$**

QUESTION 3

3 Question 2 8 / 8

✓ - 0 pts **Correct**

- 2 pts Fail to identify the region x:0->pi/2; y:0->sinx
- 2 pts Fail to apply Fubini
- 2 pts Fail to compute the integral
- 1 pts Incorrect final answer, e-1

QUESTION 4

4 Question 3 10 / 10

Part A (6 points)

✓ - 0 pts **Correct with sufficient reasoning.**

- 2 pts Minor error setting up integral. i.e, one of the bounds wrong way, or a number incorrect.
- 4 pts Major error setting up integral. i.e, the wrong region identified, or incorrect set of inequalities.
- 1 pts A minor error in calculating integral. i.e, a minor arithmetic error.
- 2 pts A major error in calculating integral.

Part B (4 points)

✓ - 0 pts **Correct with sufficient reasoning.**

- 2 pts Not identifying that the xy-projection is the same as the previous part in setting up the integral and getting it incorrect.
- 1 pts Incorrectly setting up the z bounds when using a triple integral or using the incorrect integrand.
- 1 pts Minor error in calculating the integral.

QUESTION 5

5 Question 4 8 / 8

✓ + 8 pts **Correct**

+ 1 pts Correct usage of $$- 2 \leq x \leq y \leq 0$$ in understanding $$\mathscr{D}$$
+ 1 pts Correct usage of $$x^2 + y^2 \geq 1$$ in understanding $$\mathscr{D}$$
+ 1 pts Correct drawing of $$\mathscr{D}$$
+ 1 pts Correct bounds for $$\mathscr{D}$$
+ 1 pts Correct change of variables integral setup (given the bounds from first part)
+ 1 pts Reasonably correct integration (given the integral that was set up)
+ 2 pts Correct final answer
- 2 pts No explanation

Math 32B - Lecture 4
Winter 2021
Midterm 1
Due 1/28/2021 before 10am

Sign and submit the following honor statement:

I certify on my honor that I have neither given nor received any help, or used any non-permitted resources, while completing this evaluation.

Signed:	Matthew G

Print name:	Matthew Guan

This exam contains 6 pages (including this cover page) and 4 problems. There are a total of 30 points available.

- Attempt all questions.
- Solutions must be uploaded to Gradescope before 10am Pacific Time on January 28$^{\text{th}}$.
  - Include extra pages as you need them.
  - You may complete the problems on a printout of this exam, blank paper, or a tablet/iPad.
  - If you handwrite your solutions, please make sure your scan is clearly legible.
  - Solutions should be written clearly, in full English sentences, defining all variables, showing all working, and giving units where appropriate.
- The work submitted must be entirely your own: you may not collaborate or work with anyone else to complete the exam.
- This exam is open book. You may use your notes, the textbook, and any online resource that does not involve interaction with another person.
- **Posting problems to online forums or "tutoring" websites counts as interaction with another person so is strictly forbidden.**

1 Honor statement 0 / 0

✓ + 0 pts Complete

1. (4 points) Let $\mathscr{W} = [0,1] \times [1,2] \times [2,3]$. Find

$$\iiint_{\mathscr{W}} xyz\, dV$$

our domain of integration, W, is a box with dimensions 1×1×1.
we express $\iiint_W xyz\,dV$ as an iterated integral by Fubini's Theorem.

Thus, $\iiint_W xyz\,dV = \int_2^3 \int_1^2 \int_0^1 xyz\,dx\,dy\,dz$

$= \int_2^3 \int_1^2 \left[\frac{1}{2}x^2yz\right]_0^1 dy\,dz$

$= \int_2^3 \int_1^2 \frac{1}{2}yz\,dy\,dz$

$= \int_2^3 \left(\frac{1}{4}zy^2\right)\Big|_1^2 dz$

$= \int_2^3 \frac{3}{4}z\,dz$

$= \left[\frac{3}{8}z^2\right]_2^3$

$= \frac{3}{8}(9-4) = 15/8$

So $\iiint_W xyz\,dV = \boxed{15/8}$

2 Question 1 4 / 4

- ✓ **+ 1 pts** Correct bounds for $$x$$
- ✓ **+ 1 pts** Correct bounds for $$y$$
- ✓ **+ 1 pts** Correct bounds for $$z$$
- ✓ **+ 1 pts** Correct answer of $$\frac{15}8$$

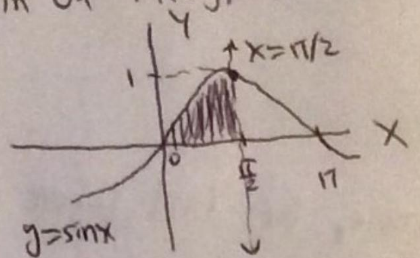
2. (8 points) Evaluate

$$\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} e^{\cos(x)}\,dx\,dy.$$

You should assume that $\sin^{-1}(y)$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(*__Hint:__ At some point in your solution, it might be useful to use the substitution $u = \cos x$.*)

we wish to reverse the order of integration. let D be the domain of integration. Sketch D. D is $\{0 \le y \le 1, \sin^{-1}(y) \le x \le \frac{\pi}{2}\}$



*we have $x = \sin^{-1}(y)$, $x = \pi/2$
so $y = \sin x$, $x = \pi/2$, will be graphed.

we notice that D is vertically simple, and can be written as D': $\{(x,y): 0 \le x \le \pi/2, 0 \le y \le \sin x\}$.

Thus, $\displaystyle\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} e^{\cos x}\,dx\,dy = \int_0^{\pi/2}\int_0^{\sin x} e^{\cos x}\,dy\,dx$

$= \displaystyle\int_0^{\pi/2} \left[y e^{\cos x} \right]_0^{\sin x} dx$

$= \displaystyle\int_0^{\pi/2} \sin x\, e^{\cos x}\, dx$

we u-substitute, let $u = \cos x$, $du = -\sin x\,dx$.

So, $\displaystyle\int_0^{\pi/2} \sin x\, e^{\cos x}\,dx = \int_1^0 -e^u\,du$

$= \left[e^u\right]_0^1$

$= e - 1.$

Thus, $\displaystyle\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} e^{\cos x}\,dx\,dy = \boxed{e-1}$.

3 Question 2 **8 / 8**

✓ **- 0 pts** **Correct**

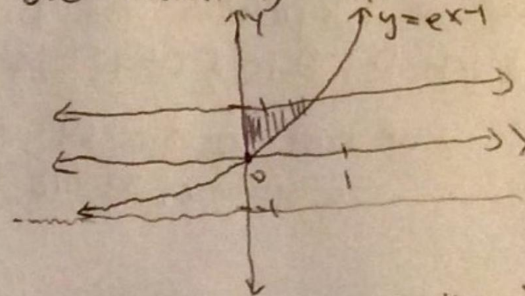
- **- 2 pts** Fail to identify the region x:0->pi/2; y:0->sinx
- **- 2 pts** Fail to apply Fubini
- **- 2 pts** Fail to compute the integral
- **- 1 pts** Incorrect final answer, e-1

3. (10 points)

(a) Let $\mathscr{D}$ be the region in the (x, y)-plane bounded by $x = 0$, $y = e^x - 1$, and $y = 1$. Use a double integral to compute the area of $\mathscr{D}$.

(b) Let $\mathscr{W}$ be the 3d region above $\mathscr{D}$ and below the surface $z = 1 - y$. Use a triple integral to compute the volume of $\mathscr{W}$.

3. a). We sketch D in the xy-plane.



We note that D is vertically simple, we find where $y=e^x-1$ and $y=1$ intersect.

$e^x-1=1 \to x=\ln 2$.

Thus, D is expressed as $\{(x,y): 0 \le x \le \ln 2, e^x-1 \le y \le 1\}$.

So, the area $(D) = \iint_D 1\,dA$. This equals $\int_0^{\ln 2}\int_{e^x-1}^{1} dy\,dx$ by Fubini's Th.

$= \int_0^{\ln 2} [y]_{e^x-1}^{1}\,dx$

$= \int_0^{\ln 2} (1-e^x+1)\,dx$

$= \int_0^{\ln 2} (2-e^x)\,dx$

$= [2x-e^x]_0^{\ln 2}$

$= (2\ln 2 - e^{\ln 2}) - (0-e^0)$

$= 2\ln 2 - 2 + 1$

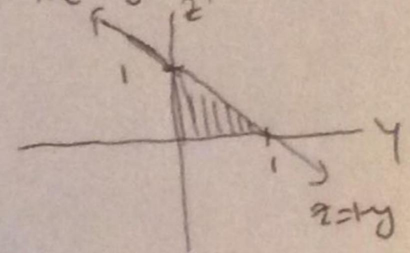
$= 2\ln 2 - 1$.

The area of D is $\boxed{2\ln(2)-1}$.

see next page for 3(b) →

3 b). The volume of W is the triple integral $\iiint_W 1\,dV$.

From part (a), we have the xy-projection of W, D. Now, sketch the yz-projection.



* From this sketch, we see that W is z-simple, with $z=0$ and $z=1-y$ being the bounding functions z_1 & z_2 of W. So, $0 \le z \le 1-y$.

From part (a), D is a vertically simple region defined by $\{(x,y): 0 \le x \le \ln(2),\ e^x-1 \le y \le 1\}$.

Thus, $\iiint_W 1\,dV = \iint_D \left(\int_0^{1-y} dz\right) dA = \int_0^{\ln 2}\int_{e^x-1}^{1}\int_0^{1-y} dz\,dy\,dx$

$= \int_0^{\ln 2}\int_{e^x-1}^{1} (1-y)\,dy\,dx$

$= \int_0^{\ln 2} \left[y - \frac{1}{2}y^2\right]_{e^x-1}^{1} dx$

$= \int_0^{\ln 2} (1-\frac{1}{2}) - ((e^x-1) - \frac{1}{2}(e^x-1)^2)\,dx.$

$= \int_0^{\ln 2} (\frac{1}{2} - e^x + 1 + \frac{1}{2}e^{2x} - e^x + \frac{1}{2})\,dx$

$= \int_0^{\ln 2} (2 - 2e^x + \frac{1}{2}e^{2x})\,dx.$

$= \left[2x - 2e^x + \frac{1}{4}e^{2x}\right]_0^{\ln 2}$

$= (2\ln 2 - 2e^{\ln 2} + \frac{1}{4}e^{2\ln 2}) - (0 - 2 + \frac{1}{4})$

$= 2\ln 2 - 4 + 1 + 2 - 1/4$

$= 2\ln 2 - \frac{5}{4}$

The volume of W is $\boxed{2\ln(2) - \frac{5}{4}.}$

4 Question 3 **10 / 10**

Part A (6 points)

✓ **- 0 pts** **Correct with sufficient reasoning.**

- 2 pts Minor error setting up integral. i.e, one of the bounds wrong way, or a number incorrect.

- 4 pts Major error setting up integral. i.e, the wrong region identified, or incorrect set of inequalities.

- 1 pts A minor error in calculating integral. i.e, a minor arithmetic error.

- 2 pts A major error in calculating integral.

Part B (4 points)

✓ **- 0 pts** **Correct with sufficient reasoning.**

- 2 pts Not identifying that the xy-projection is the same as the previous part in setting up the integral and getting it incorrect.

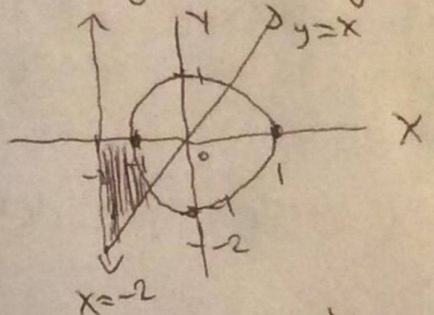
- 1 pts Incorrectly setting up the z bounds when using a triple integral or using the incorrect integrand.

- 1 pts Minor error in calculating the integral.

4. (8 points) Let $\mathscr{D}$ be the region where $-2 \le x \le y \le 0$ and $x^2 + y^2 \ge 1$. Evaluate

$$\iint_{\mathscr{D}} (x^2+y^2)^{-\frac{3}{2}}\, dA.$$

We start by sketching out D. We have $x \ge -2$, $y \ge x$, and $y \le 0$ and $x^2+y^2 \ge 1$.



* We see that D is radially simple. Translate the equations to polar.

$x^2+y^2=1 \to r=1$.

$y=x \to r\sin\theta = r\cos\theta \to \tan\theta = 1$. $\theta = \frac{5\pi}{4}$.

$x=-2 \to r\cos\theta = -2. \to r = -2/\cos\theta$.

We see that θ is fixed by the lines $y=0$ and $y=x$, so the bounds for θ are $\{\pi \le \theta \le \frac{5\pi}{4}\}$. We see that r ranges from 1 to $-2\sec\theta$.

Thus, $\iint_D (x^2+y^2)^{-3/2}\, dA = \int_{\pi}^{5\pi/4} \int_1^{-2\sec\theta} r \cdot r^{-3}\, dr\, d\theta$.

$= \int_{\pi}^{5\pi/4} \int_1^{-2\sec\theta} r^{-2}\, dr\, d\theta$.

$= \int_{\pi}^{5\pi/4} \left[-\frac{1}{r}\right]_1^{-2\sec\theta} d\theta$.

$= \int_{\pi}^{5\pi/4} \left(\frac{1}{2}\cos\theta + 1\right) d\theta$.

$= \left[\frac{1}{2}\sin\theta + \theta\right]_{\pi}^{5\pi/4}$.

$= \frac{1}{2}\sin\left(\frac{5\pi}{4}\right) + \frac{5\pi}{4} - \pi$.

$= -\frac{\sqrt{2}}{4} + \frac{\pi}{4} = \frac{\pi - \sqrt{2}}{4}$.

Thus, $\iint_D (x^2+y^2)^{-3/2}\, dA = \boxed{\frac{\pi-\sqrt{2}}{4}}$.

5 Question 4 **8 / 8**

✓ **+ 8 pts** **Correct**

- **+ 1 pts** Correct usage of $-2 \leq x \leq y \leq 0$ in understanding $\mathscr{D}$
- **+ 1 pts** Correct usage of $x^2 + y^2 \geq 1$ in understanding $\mathscr{D}$
- **+ 1 pts** Correct drawing of $\mathscr{D}$
- **+ 1 pts** Correct bounds for $\mathscr{D}$
- **+ 1 pts** Correct change of variables integral setup (given the bounds from first part)
- **+ 1 pts** Reasonably correct integration (given the integral that was set up)
- **+ 2 pts** Correct final answer
- **- 2 pts** No explanation