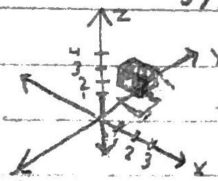
1. let $W = [0,1] \times [1,2] \times [2,3]$ and find $\iiint_W xyz\,dV$

Start by sketching the region W:



because the region W is a box where $0 \le x \le 1$, $1 \le y \le 2$, and $2 \le z \le 3$, we can apply Fubini's theorem to evaluate the triple integral.

according to Fubini's theorem, $\iiint_W xyz\,dV = \int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} xyz\,dz\,dy\,dx$

so $\iiint_W xyz\,dV = \int_0^1\int_1^2\int_2^3 xyz\,dz\,dy\,dx$

evaluating the integral: $\iiint_W xyz\,dV = \int_0^1\int_1^2 \left[\frac{xyz^2}{2}\right]_2^3 dy\,dx = \int_0^1\int_1^2 \frac{xy(3)^2}{2} - \frac{xy(2)^2}{2}\,dy\,dx$

$= \int_0^1\int_1^2 \frac{9xy}{2} - \frac{4xy}{2}\,dy\,dx = \int_0^1\int_1^2 \frac{5xy}{2}\,dy\,dx = \int_0^1 \left[\frac{5xy^2}{2\cdot 2}\right]_1^2 dx$

$= \int_0^1 \frac{5x(2)^2}{4} - \frac{5x(1)^2}{4}\,dx = \int_0^1 \frac{20x}{4} - \frac{5x}{4}\,dx = \int_0^1 \frac{15x}{4}\,dx$

$= \left[\frac{15x^2}{4\cdot 2}\right]_0^1 = \frac{15(1)^2}{8} - \frac{15(0)^2}{8} = \frac{15}{8}$

$\iiint_W xyz\,dV = \frac{15}{8}$

2 Question 1 **4 / 4**

- ✓ **+ 1 pts** Correct bounds for $$x$$
- ✓ **+ 1 pts** Correct bounds for $$y$$
- ✓ **+ 1 pts** Correct bounds for $$z$$
- ✓ **+ 1 pts** Correct answer of $$\frac{15}8$$

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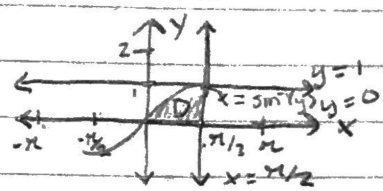
2. evaluate $\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} e^{\cos(x)}\,dx\,dy$

Start by sketching the domain D as defined in the integral above, where $\sin^{-1}(y) \le x \le \frac{\pi}{2}$ and $0 \le y \le 1$. (and $\sin^{-1}(y)$ goes from $-\pi/2$ to $\pi/2$)

$x = \sin^{-1}(y) \rightarrow \sin x = \sin(\sin^{-1}(y))$

$\sin x = y$



This region D is currently being used as a horizontally simple region. However, it can be changed to a vertically simple region, which would make the integral much nicer, by switching the order of integration.

Describing D as a vertically simple region: $D = \{(x,y): 0 \le x \le \frac{\pi}{2},\ 0 \le y \le \sin x\}$

as D is now a vertically simple region, and we know that generally

$\iint_D f(x,y)\,dA = \int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$ (and in our case $f(x,y) = e^{\cos(x)}$)

we can say that $\iint_D e^{\cos(x)}\,dA = \int_0^{\pi/2}\int_0^{\sin x} e^{\cos(x)}\,dy\,dx$

and because D describes the same region as in the original integral,

$\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} e^{\cos(x)}\,dx\,dy = \int_0^{\pi/2}\int_0^{\sin x} e^{\cos(x)}\,dy\,dx$

$= \int_0^{\pi/2} \left[y e^{\cos(x)} \right]_0^{\sin x} dx = \int_0^{\pi/2} \sin x\, e^{\cos x} - 0\,dx$

$= \int_0^{\pi/2} \sin x\, e^{\cos x}\,dx$ let $u = \cos x$, so $du = -\sin x\,dx$

so $= -\int_{x=0}^{x=\pi/2} e^u\,du = -\left[e^u\right]_{x=0}^{x=\pi/2} = -\left[e^{\cos x}\right]_0^{\pi/2}$

$= -(e^{\cos(\pi/2)} - e^{\cos(0)}) = -e^0 + e^1 = -1 + e = e - 1$

so $\int_0^1 \int_{\sin^{-1}(y)}^{\pi/2} e^{\cos(x)}\,dx\,dy = e - 1$

3 Question 2 **8 / 8**

✓ **- 0 pts** **Correct**

- 2 pts Fail to identify the region x:0->pi/2; y:0->sinx

- 2 pts Fail to apply Fubini

- 2 pts Fail to compute the integral

- 1 pts Incorrect final answer, e-1

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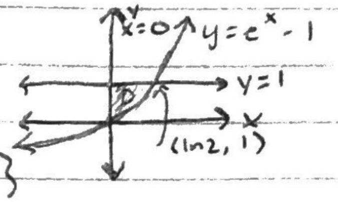
3\. a) let D be the region in the (x,y) plane bounded by $x=0$, $y=e^x-1$ and $y=1$. Use a double integral to compute the area of D.

Start by sketching the region D:



Intersection point of $y=e^x-1$ and $y=1$:
$1=e^x-1 \quad 2=e^x$
$\ln 2=\ln e^x=x$
so at $(\ln 2, 1)$

From the graph, $D=\{(x,y):\ 0\le x\le \ln 2,\ e^x-1\le y\le 1\}$

so because D is vertically simple, $\iint_D f(x,y)\,dA=\int_{x_1}^{x_2}\int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$

since we are finding the area of D, our $f(x,y)=1$

so $\text{Area}(D)=\int_0^{\ln 2}\int_{e^x-1}^{1} 1\,dy\,dx=\int_0^{\ln 2}[y]_{e^x-1}^{1}\,dx=\int_0^{\ln 2} 1-e^x+1\,dx$

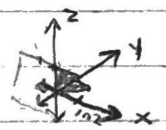
$=\int_0^{\ln 2} 2-e^x\,dx=[2x-e^x]_0^{\ln 2}=2\ln 2-e^{\ln 2}-2(0)+e^0$

$=2\ln 2-2+1=2\ln 2-1$

so $\text{Area}(D)=2\ln 2-1$

b) let W be the 3D region above D and below the surface $z=1-y$. use a triple integral to compute the volume of W

making a sketch of W:



$0\le z\le 1-y$ and the bounds for D don't change because at $z=0$ $y=1$ which is included in D.

So the region $W=\{(x,y,z):\ 0\le x\le \ln 2,\ e^x-1\le y\le 1,\ 0\le z\le 1-y\}$

this region is z simple, as it is contained in a domain D and z is bounded by functions of (x,y). because of this, we can say $\text{Volume}(W)=\iiint_W 1\,dV$

and $\text{Volume}(W)=\iiint_W 1\,dV=\iint_D\left(\int_0^{1-y} dz\right)dA=\int_0^{\ln 2}\int_{e^x-1}^{1}\int_0^{1-y} 1\,dz\,dy\,dx$

$=\int_0^{\ln 2}\int_{e^x-1}^{1}[z]_0^{1-y}\,dy\,dx=\int_0^{\ln 2}\int_{e^x-1}^{1} 1-y\,dy\,dx$

$=\int_0^{\ln 2}\left[y-\frac{y^2}{2}\right]_{e^x-1}^{1}dx=\int_0^{\ln 2} 1-\frac{1^2}{2}-\left(e^x-1-\frac{(e^x-1)^2}{2}\right)dx$

$=\int_0^{\ln 2}\frac{1}{2}-e^x+1+\frac{e^{2x}-2e^x+1}{2}\,dx=\int_0^{\ln 2}\frac{3}{2}-e^x+\frac{e^{2x}}{2}-e^x+\frac{1}{2}\,dx$

$=\int_0^{\ln 2} 2-2e^x+\frac{e^{2x}}{2}\,dx=\left[2x-2e^x+\frac{e^{2x}}{2\cdot 2}\right]_0^{\ln 2}=2\ln 2-2e^{\ln 2}+\frac{e^{2\ln 2}}{4}-\left(0-2e^0+\frac{e^0}{4}\right)$

$=2\ln 2-(2\cdot 2)+\frac{2\cdot 2}{4}+2(1)-\frac{1}{4}=2\ln 2-4+1+2-\frac{1}{4}$

$=2\ln 2-\frac{5}{4}$

$\text{Volume}(W)=2\ln 2-\frac{5}{4}$

4 Question 3 **10 / 10**

Part A (6 points)

✓ **- 0 pts** **Correct with sufficient reasoning.**

- 2 pts Minor error setting up integral. i.e, one of the bounds wrong way, or a number incorrect.

- 4 pts Major error setting up integral. i.e, the wrong region identified, or incorrect set of inequalities.

- 1 pts A minor error in calculating integral. i.e, a minor arithmetic error.

- 2 pts A major error in calculating integral.

Part B (4 points)

✓ **- 0 pts** **Correct with sufficient reasoning.**

- 2 pts Not identifying that the xy-projection is the same as the previous part in setting up the integral and getting it incorrect.

- 1 pts Incorrectly setting up the z bounds when using a triple integral or using the incorrect integrand.

- 1 pts Minor error in calculating the integral.

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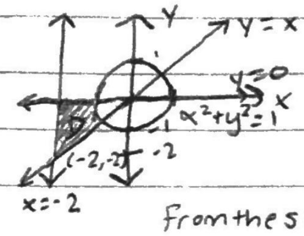
4. Let D be the region where $-2 \le x \le y \le 0$ and $x^2+y^2 \ge 1$

evaluate $\iint_D (x^2+y^2)^{-3/2}\, dA$

Start by making a sketch of D:



$x^2+y^2=1$

$x^2 = 1-y^2$

$x = -\sqrt{1-y^2}$

From the sketch, $D = \{(x,y): -2 \le x \le -\sqrt{1-y^2}$

$x \le y \le 0\}$

this integral will be easier to evaluate in polar coordinates. to convert the domain to polar coordinates, we start by determining r. r's lower bound is 1, the radius of the circle $x^2+y^2 \ge 1$.

r's upper bound is determined by $x=-2$, or $r\cos\theta = -2$, or $r = \frac{-2}{\cos\theta}$.

for θ's bounds, its lower bound is clearly π $(\tan^{-1}(\frac{0}{1}) = \pi)$

and it's upper bound can be found using the point $(x,y) = (-2,-2)$

where $\theta = \tan^{-1}(\frac{-2}{-2}) = \tan^{-1}(1) = \frac{5\pi}{4}$

so if we let D_0 be the region D represented in polar coordinates,

$D_0 = \{(r,\theta): 1 \le r \le \frac{-2}{\cos\theta},\ \pi \le \theta \le \frac{5\pi}{4}\}$

this region is radially simple, so $\iint_D f(x,y)\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$

in this case, $f(x,y) = (x^2+y^2)^{-3/2}$ and $f(r\cos\theta, r\sin\theta) = (r^2\cos^2\theta + r^2\sin^2\theta)^{-3/2}$

$f(r\cos\theta, r\sin\theta) = (r^2)^{-3/2} = r^{-3}$

so $\iint_D (x^2+y^2)^{-3/2}\,dA = \int_{\pi}^{\frac{5\pi}{4}}\int_1^{\frac{-2}{\cos\theta}} r^{-3}\, r\,dr\,d\theta = \int_{\pi}^{5\pi/4}\int_1^{\frac{-2}{\cos\theta}} r^{-2}\,dr\,d\theta$

$= \int_{\pi}^{5\pi/4} \left[-r^{-1}\right]_1^{\frac{-2}{\cos\theta}} d\theta = \int_{\pi}^{5\pi/4} \left(\frac{2}{\cos\theta}\right)^{-1} - (-1)\,d\theta = \int_{\pi}^{5\pi/4} \frac{\cos\theta}{2} + 1\,d\theta$

$= \left[-\frac{\sin\theta}{2} + \theta\right]_{\pi}^{\frac{5\pi}{4}} = \frac{-\sin(\frac{5\pi}{4})}{2} + \frac{5\pi}{4} - \left(-\frac{\sin\pi}{2} + \pi\right)$

$= \frac{-(-\sqrt{2}/2)}{2} + \frac{5\pi}{4} + \frac{0}{2} - \pi = \frac{\sqrt{2}}{4} + \frac{5\pi}{4} - \pi = \frac{\sqrt{2}+\pi}{4}$

$\iint_D (x^2+y^2)^{-3/2}\,dA = \frac{\sqrt{2}+\pi}{4}$

5 Question 4 **6 / 8**

+ **8 pts** Correct

✓ + **1 pts Correct usage of $$- 2 \leq x \leq y \leq 0$$ in understanding $$\mathscr{D}$$**

✓ + **1 pts Correct usage of $$x^2 + y^2 \geq 1$$ in understanding $$\mathscr{D}$$**

✓ + **1 pts Correct drawing of $$\mathscr{D}$$**

✓ + **1 pts Correct bounds for $$\mathscr{D}$$**

✓ + **1 pts Correct change of variables integral setup (given the bounds from first part)**

✓ + **1 pts Reasonably correct integration (given the integral that was set up)**

+ **2 pts** Correct final answer

- **2 pts** No explanation