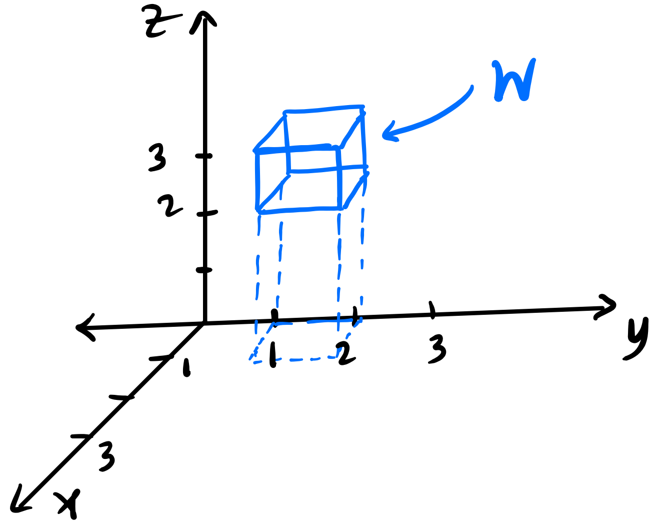
1. (4 points) Let $\mathscr{W} = \overset{x}{[0,1]} \times \overset{y}{[1,2]} \times \overset{z}{[2,3]}$. Find

$$\iiint_{\mathscr{W}} xyz\, dV$$



$$\iiint_W xyz\, dz = \int_0^1 \int_1^2 \int_2^3 xyz\, dz\, dy\, dx$$

$$= \int_0^1 x\, dx \int_1^2 y\, dy \int_2^3 z\, dz$$

$$= \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_1^2 \left[\frac{z^2}{2}\right]_2^3$$

$$= \left(\frac{1}{2} - 0\right)\left(\frac{4}{2} - \frac{1}{2}\right)\left(\frac{9}{2} - \frac{4}{2}\right)$$

$$= \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$$

$$= \boxed{\frac{15}{8}}$$

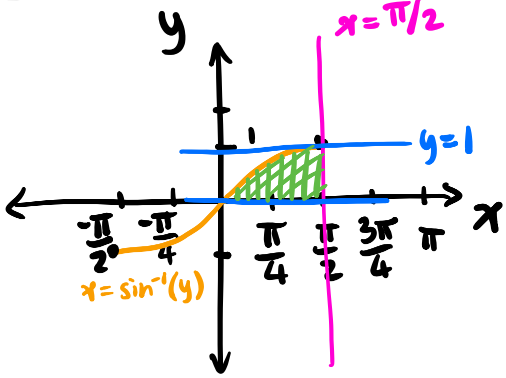
2. (8 points) Evaluate

$$\int_0^1 \int_{\sin^{-1}(y)}^{\frac{\pi}{2}} e^{\cos(x)}\, dx\, dy.$$

You should assume that $\sin^{-1}(y)$ has range $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

(*Hint:* *At some point in your solution, it might be useful to use the substitution* $u = \cos x$.)

Step 1: draw the domain



Step 2: swap order

$0 \le x \le \frac{\pi}{2}$

$0 \le y \le \sin x$

Step 3: compute

$$\int_0^{\pi/2} \int_0^{\sin x} e^{\cos x}\, dy\, dx = \int_0^{\pi/2} \left[y e^{\cos x} \right]_0^{\sin x} dx$$

$$= \int_0^{\pi/2} \sin x\, e^{\cos x}\, dx \Rightarrow \begin{Bmatrix} u = \cos x \\ du = -\sin x\, dx \end{Bmatrix}$$

$$= -\int_0^{\pi/2} e^u\, du = -\left[e^{\cos x} \right]_0^{\pi/2} = -\left(e^{\cos \frac{\pi}{2}} - e^{\cos 0} \right)$$

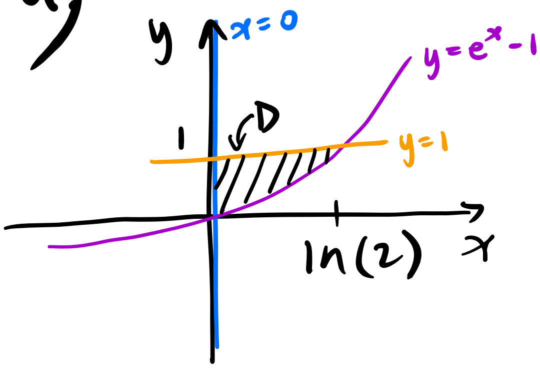
$$= -(e^0 - e^1) = -(1-e) = \boxed{e-1}$$

3. (10 points)

(a) Let $\mathscr{D}$ be the region in the (x, y)-plane bounded by $x = 0$, $y = e^x - 1$, and $y = 1$. Use a double integral to compute the area of $\mathscr{D}$.

(b) Let $\mathscr{W}$ be the 3d region above $\mathscr{D}$ and below the surface $z = 1 - y$. Use a triple integral to compute the volume of $\mathscr{W}$.

a) Step 1: draw the region



Step 2: find intersection point for x upper bound

$e^x - 1 = 1$

$e^x = 2$

$\therefore x = \ln 2$

Step 3: find the domain

$\therefore D = \{(x,y) : 0 \le x \le \ln(2),\ e^x - 1 \le y \le 1\}$

Step 4: compute

$$\therefore \iint_D dA = \int_0^{\ln(2)} \int_{e^x-1}^{1} dy\, dx = \int_0^{\ln(2)} \left[y\right]_{e^x-1}^{1} dx$$

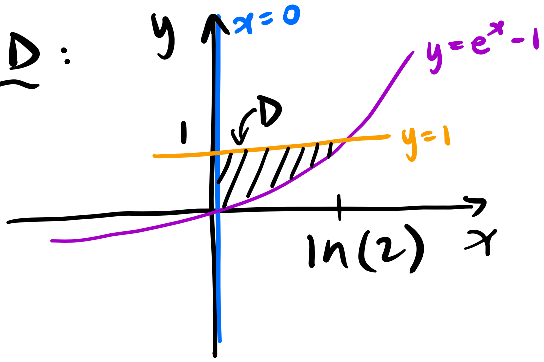
$$= \int_0^{\ln(2)} 1 - (e^x - 1)\, dx = \int_0^{\ln(2)} 1 - e^x + 1\, dx = \int_0^{\ln(2)} 2 - e^x\, dx$$

$$= \left[2x - e^x\right]_0^{\ln(2)} = 2\ln(2) - e^{\ln(2)} - 2(0) - (-e^0)$$

$$= 2\ln(2) - 2 - 0 + 1 = \boxed{2\ln(2) - 1}$$

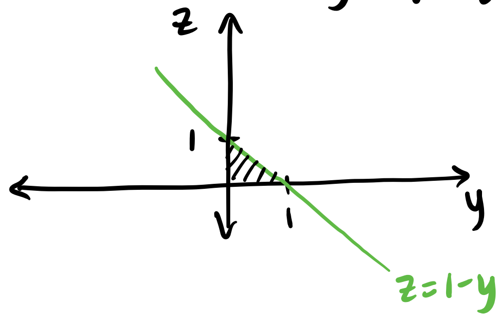
b.) W: region D + below $z = 1-y$

region D :



$\therefore$ based on the region D, $0 \le y \le 1$

Step 1 : draw yz projection



Step 2: determine z bounds based on yz projection, $0 \le z \le 1-y$

Step 3: determine domain

$$D = \{(x,y,z) : 0 \le x \le \ln(2),\ e^x - 1 \le y \le 1,\ 0 \le z \le 1-y\}$$

Step 4 : compute

$$\int_0^{\ln(2)} \int_{e^x-1}^{1} \int_0^{1-y} dz\,dy\,dx = \int_0^{\ln(2)} \int_{e^x-1}^{1} 1-y \; dy\,dx = \int_0^{\ln(2)} \left[y - \frac{y^2}{2}\right]_{e^x-1}^{1} dx$$

$$= \int_0^{\ln(2)} 1 - \frac{1}{2} - (e^x - 1) + \frac{(e^x-1)^2}{2}\, dx = \int_0^{\ln(2)} \frac{1}{2} - e^x + 1 + \frac{(e^{2x} - 2e^x + 1)}{2}\, dx$$

$$= \frac{1}{2}\int_0^{\ln(2)} 1 - 2e^x + 2 + e^{2x} - 2e^x + 1\, dx = \frac{1}{2}\int_0^{\ln(2)} 4 - 4e^x + e^{2x}\, dx$$

$$= \frac{1}{2}\left[4x - 4e^x + \frac{1}{2}e^{2x}\right]_0^{\ln(2)}$$

$$= \frac{1}{2}\left(4\ln(2) - 4(2) + \frac{1}{2}e^{2\ln(2)} - 4(0) + 4(1) - \frac{1}{2}(1)\right)$$

$$= \frac{1}{2}\left(4\ln(2) - 8 + \frac{4}{2} + 4 - \frac{1}{2}\right) = \frac{1}{2}\left(4\ln(2) - \frac{5}{2}\right)$$

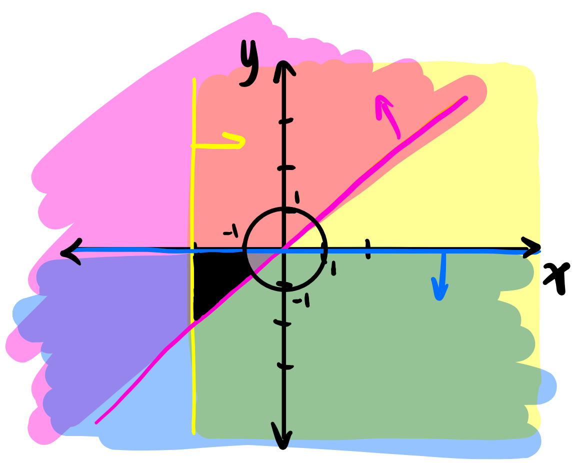
$$= \boxed{2\ln(2) - \frac{5}{4}}$$

4. (8 points) Let $\mathscr{D}$ be the region where $-2 \le x \le y \le 0$ and $x^2 + y^2 \ge 1$. Evaluate

$$\iint_{\mathscr{D}} (x^2+y^2)^{-\frac{3}{2}}\, dA.$$

$x \ge -2$ $\quad y \ge x$ $\quad y \le 0$ $\quad x^2+y^2 \ge 1$

Step 1: draw $\mathscr{D}$



Step 2: analyze bounds

① $x^2+y^2 \ge 1 \xrightarrow[\text{to polar}]{\text{convert}} r^2 \ge 1 \therefore r \ge 1$

② $y \le 0 \xrightarrow[\text{to polar}]{\text{convert}} r\sin\theta \le 0 \therefore \sin\theta \le 0 \therefore \theta$ must be in 3^{rd} or 4^{th} quadrant

③ $y \ge x \xrightarrow[\text{to polar}]{\text{convert}} r\sin\theta \ge r\cos\theta \therefore \sin\theta \ge \cos\theta \therefore \theta$ must be between $\frac{\pi}{4}$ and $\frac{5\pi}{4}$

*based on ② and ③, the bounds of θ must be $\pi \le \theta \le \frac{5\pi}{4}$

④ $x \ge -2 \xrightarrow[\text{to polar}]{\text{convert}} r\cos\theta \ge -2$

$\therefore r \ge \frac{-2}{\cos\theta}$, but since $\pi \le \theta \le \frac{5\pi}{4}$, $\cos\theta < 0$

So, $r \le \frac{2}{\cos\theta}$

Step 3: determine domain in polar coordinates

$$D = \left\{ (r,\theta) : \pi \le \theta \le \frac{5\pi}{4},\ 1 \le r \le \frac{2}{\cos\theta} \right\}$$

Step 4: convert function to polar : $(x^2+y^2)^{-3/2} \Rightarrow (r^2)^{-3/2} = r^{-3}$

Step 5: compute

$$\int_{\pi}^{5\pi/4}\int_{1}^{2/\cos\theta} r^{-3} \cdot r\,dr\,d\theta = \int_{\pi}^{5\pi/4}\int_{1}^{2/\cos\theta} r^{-2}\,dr\,d\theta = \int_{\pi}^{5\pi/4}\left[-\frac{1}{r}\right]_{1}^{2/\cos\theta} d\theta$$

$$= \int_{\pi}^{5\pi/4} -\frac{\cos\theta}{2} + 1\,d\theta = -\frac{1}{2}\int_{\pi}^{5\pi/4} \cos\theta - 2\,d\theta = -\frac{1}{2}\left[\sin\theta - 2\theta\right]_{\pi}^{5\pi/4}$$

$$= -\frac{1}{2}\left(\sin\frac{5\pi}{4} - \frac{10\pi}{4} - \sin\pi + 2\pi\right) = -\frac{1}{2}\left(-\frac{\sqrt{2}}{2} - \frac{5\pi}{2} + 2\pi\right)$$

$$= -\frac{1}{2}\left(-\frac{\sqrt{2}}{2} - \frac{\pi}{2}\right) = \boxed{\frac{\sqrt{2}}{4} + \frac{\pi}{4}}$$