Math 32B Midterm 1R

MINGYANG ZHANG

TOTAL POINTS

39 / 40

QUESTION 1

1 Product rule 5 / 5

- ✓ + **2 pts** **Correct expression for curl operator in components**
- ✓ + **1 pts** **Correct expression for $$\nabla f$$**
- ✓ + **1 pts** **Correct application of the product rule for partial derivatives**
- ✓ + **1 pts** **Solution clearly explained**
- \+ **0 pts** No credit due

QUESTION 2

2 Line integral 7 / 8

- ✓ + **4 pts** **Parametrized curve correctly, except possibly orientation**
- \+ **2 pts** Correct orientation of curve/integral
- ✓ + **2 pts** **Successful computation of the correct integral, except possibly orientation/sign**
- ✓ + **1 pts** **Clearly explaining your solution**
- \+ **1 pts** Bonus: sketch of curve (with or without orientation)
- \+ **0 pts** No points

QUESTION 3

3 Fundamental Theorem of Vector Line Integrals 12 / 12

- ✓ + **1 pts** **correct answer (-sqrt(2) - pi/2), correctly derived using potential function, and solution is clearly explained**
- ✓ + **9 pts** **correct potential function f(x,y,z) = sqrt(1+y^2) + cos(x-z) - z**
- ✓ + **2 pts** **integral is equal to f(pi,0,pi/2) - f(0,1,0)**
- \+ **1 pts** (incorrect) integral is equal to f(pi,0,pi/2) - f(0,1,0)
- \+ **0 pts** no points
- \+ **7 pts** partial credit for nearly correct expression for potential function
- \+ **3 pts** partial credit for some progress towards finding a potential function
- \+ **2 pts** correct expression for a parametric curve from (0,1,0) to (pi,0,pi/2) (only if no solution via potential function)
- \+ **2 pts** correct expression for a vector line integral using a parametric curve (only if no solution via potential function)
- \+ **8 pts** correct answer (-sqrt(2) - pi/2), correctly derived via a parametric curve, and solution is clearly explained
- \+ **1 pts** partial credit for incorrect expression for parametric curve (only if no solution via potential function)

QUESTION 4

4 Volume via a double integral 15 / 15

- ✓ + **1 pts** **Clearly written attempt**
- ✓ + **2 pts** **Drawing and labelling of region (4 pts)**
- ✓ + **1 pts** **Drawing and labelling of region**
- ✓ + **1 pts** **Draw/label region (deriving x-intersection points)**
- ✓ + **2 pts** **Correctly set-up integral (5 pts)**
- ✓ + **2 pts** **Correctly set-up integral**
- ✓ + **1 pts** **Correctly set-up integral**
- ✓ + **2 pts** **Showing z = 2 - x^2 is the top surface (2 pts)**
- ✓ + **2 pts** **Computation (3 pts max)**
- ✓ + **1 pts** **Computation**

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Name: Mingyang Zhang

SID: 405170429

Dis: 3E

Time Limit: 50 Minutes

Version (→)

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\text{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f\,\text{curl}\,\mathbf{F}.$$

we know $\text{curl}(f\vec{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \left\langle \frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z}, \frac{\partial(fF_1)}{\partial z} - \frac{\partial(fF_3)}{\partial x}, \frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y} \right\rangle$ ①

and $\nabla f \times \vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, \frac{\partial f}{\partial z} F_1 - \frac{\partial f}{\partial x} F_3, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right\rangle$

$f\,\text{curl}\,\vec{F} = f \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle f\frac{\partial F_3}{\partial y} - f\frac{\partial F_2}{\partial z}, f\frac{\partial F_1}{\partial z} - f\frac{\partial F_3}{\partial x}, f\frac{\partial F_2}{\partial x} - f\frac{\partial F_1}{\partial y} \right\rangle$

From ①, we know that

$\text{curl}(f\vec{F}) = \left\langle \frac{\partial f}{\partial y}F_3 + \frac{\partial F_3}{\partial y}f - \frac{\partial f}{\partial z}F_2 - \frac{\partial F_2}{\partial z}f, \frac{\partial f}{\partial z}F_1 + \frac{\partial F_1}{\partial z}f - \frac{\partial f}{\partial x}F_3 - \frac{\partial F_3}{\partial x}, \frac{\partial f}{\partial x}F_2 + \frac{\partial F_2}{\partial x}f - \frac{\partial f}{\partial y}F_1 - \frac{\partial F_1}{\partial y}f \right\rangle$

$= \left\langle \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right) + \left(f\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}f\right), \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\right) + \left(f\frac{\partial F_1}{\partial z} - f\frac{\partial F_3}{\partial x}\right), \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right) + \left(f\frac{\partial F_2}{\partial x} - f\frac{\partial F_1}{\partial y}\right) \right\rangle$

$= \nabla f \times \vec{F} + f\,\text{curl}\,\vec{F}$

2. (8 points) Let $C \subset \mathbb{R}^2$ be the part of the curve $y = x^4$ from $x = 1$ to $x = -1$. Find

$$\int_C (1 + x)\, dy - y\, dx.$$

$\int_C$ ~~but~~ we can parametrize $y = x^4$ to $r(t) = \langle t, t^4 \rangle$

$\Rightarrow$ we compute $dy/dt = 4t^3$ $dx/dt = 1$ with domain $(-1 \le t \le 1)$

Then we compute

$$\int_C (1+x)dy - y\,dx = \int_{-1}^{1} \left((1+x)\frac{dy}{dt} - y\frac{dx}{dt}\right)dt = \int_{-1}^{1} (1+t)4t^3 - t^4\, dt$$

$$= \int_{-1}^{1} 4t^3 + 3t^4\, dt = t^4 + \frac{3}{5}t^5 \Big|_{-1}^{1} = \left(1 + \frac{3}{5}\right) - \left(1 - \frac{3}{5}\right)$$

$$= 1 + \frac{3}{5} - 1 + \frac{3}{5} = \frac{6}{5}$$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle -\sin(x-z)\ ,\ \frac{y}{\sqrt{1+y^2}}\ ,\ \sin(x-z)-1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $(0,1,0)$ to $(\pi,0,\frac{\pi}{2})$.

we need to find a potential function f for $\vec{F}$ s.t. $\nabla f = \vec{F}$

therefore, $f = \int -\sin(x-z)\,dx = \cos(x-z) + C(y,z)$ for some function C with respect to y,z.

we know $\frac{\partial f}{\partial y} = \frac{y}{\sqrt{1+y^2}} \Rightarrow f = \cos(x-z) + \int \frac{y}{\sqrt{1+y^2}}\,dy + C_1(z)$

$= \cos(x-z) + \sqrt{y^2+1} - C_1(z)$ for some function C_1 with respect to z

we also know $\frac{\partial f}{\partial z} = \sin(x-z) - 1 \Rightarrow C_1(z) = -z$

$\Rightarrow f = \cos(x-z) + \sqrt{y^2+1} - z$ and $\nabla f = \vec{F}$

Therefore $\int_C \vec{F}\,d\vec{r} = f(\pi, 0, \frac{\pi}{2}) - f(0,1,0)$

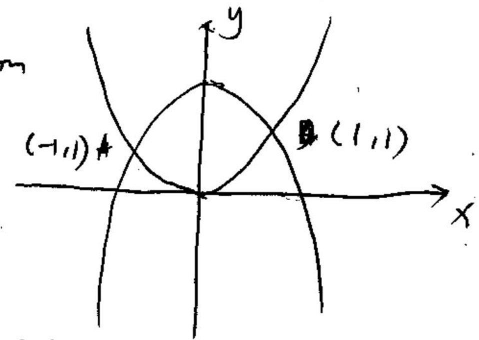
$= \cos(\pi - \frac{\pi}{2}) + 1 - \frac{\pi}{2} - (1+\sqrt{2})$

$= 1 - \frac{\pi}{2} - 1 - \sqrt{2}$

$= -\frac{\pi}{2} - \sqrt{2}$

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = x^2$, $z = 2 - x^2$.

we first draw the graph of the bottom



$\begin{cases} y = x^2 \\ y = 2-x^2 \end{cases} \Rightarrow \begin{cases} x = 1 \\ y = 1 \end{cases}$ or $\begin{cases} x = -1 \\ y = +1 \end{cases}$

$D = \{ -1 \le x \le 1 , \quad x^2 \le y \le 2-x^2 \}$

since when $-1 \le x \le 1$, $z = x^2$ is under $z = 2-x^2$, therefore

$h = 2-x^2 - x^2 = 2-2x^2$

therefore $V_W = \iint_D 2-2x^2 \, dA = \int_{-1}^{1}\int_{x^2}^{2-x^2} 2-2x^2 \, dy\, dx$

$= \int_{-1}^{1} (2-2x^2)\, y \Big|_{x^2}^{2-x^2} dx = \int_{-1}^{1} (2-2x^2)^2 dx = \int_{-1}^{1} 4 - 8x^2 + 4x^4 dx$

$= 4x - \frac{8}{3}x^3 + \frac{4}{5}x^5 \Big|_{-1}^{1} = (4 - \frac{8}{3} + \frac{4}{5}) - (-4 + \frac{8}{3} - \frac{4}{5})$

$= \frac{64}{15}$