Math 32B Midterm 1R

FRANK XING

TOTAL POINTS

37 / 40

QUESTION 1

1 Product rule 4 / 5

✓ **+ 2 pts Correct expression for curl operator in components**

✓ **+ 1 pts Correct expression for $$\nabla f$$**

+ **1 pts** Correct application of the product rule for partial derivatives

✓ **+ 1 pts Solution clearly explained**

+ **0 pts** No credit due

How do you get your second equality using the chain rule? I don't understand.

QUESTION 2

2 Line integral 7 / 8

✓ **+ 4 pts Parametrized curve correctly, except possibly orientation**

✓ **+ 2 pts Correct orientation of curve/integral**

+ **2 pts** Successful computation of the correct integral, except possibly orientation/sign

✓ **+ 1 pts Clearly explaining your solution**

+ **1 pts** Bonus: sketch of curve (with or without orientation)

+ **0 pts** No points

Symmetry doesn't work like that.

QUESTION 3

3 Fundamental Theorem of Vector Line Integrals 12 / 12

✓ **+ 1 pts correct answer (-sqrt(2) - pi/2), correctly derived using potential function, and solution is clearly explained**

✓ **+ 9 pts correct potential function f(x,y,z) = sqrt(1+y^2) + cos(x-z) - z**

✓ **+ 2 pts integral is equal to f(pi,0,pi/2) - f(0,1,0)**

+ **1 pts** (incorrect) integral is equal to f(pi,0,pi/2) - f(0,1,0)

+ **0 pts** no points

+ **7 pts** partial credit for nearly correct expression for potential function

+ **3 pts** partial credit for some progress towards finding a potential function

+ **2 pts** correct expression for a parametric curve from (0,1,0) to (pi,0,pi/2) (only if no solution via potential function)

+ **2 pts** correct expression for a vector line integral using a parametric curve (only if no solution via potential function)

+ **8 pts** correct answer (-sqrt(2) - pi/2), correctly derived via a parametric curve, and solution is clearly explained

+ **1 pts** partial credit for incorrect expression for parametric curve (only if no solution via potential function)

QUESTION 4

4 Volume via a double integral 14 / 15

✓ **+ 1 pts Clearly written attempt**

✓ **+ 2 pts Drawing and labelling of region (4 pts)**

✓ **+ 1 pts Drawing and labelling of region**

+ **1 pts** Draw/label region (deriving x-intersection points)

✓ **+ 2 pts Correctly set-up integral (5 pts)**

✓ **+ 2 pts Correctly set-up integral**

✓ **+ 1 pts Correctly set-up integral**

✓ **+ 2 pts Showing z = 2 - x^2 is the top surface (2 pts)**

✓ **+ 2 pts Computation (3 pts max)**

✓ **+ 1 pts Computation**

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Name: Frank Xing

SID: 905-164-685

Time Limit: 50 Minutes

Version $\rightarrow$

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

$\frac{df}{dx}$, dy

$\operatorname{curl}\cdot(fF) = \nabla f \times F + f \operatorname{curl} F$

curl(fF) can be thought as $\nabla \times fF$.

$$\begin{pmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ f & f & f \end{pmatrix} \quad \begin{pmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ F_1 & F_2 & F_3 \end{pmatrix}$$

$\operatorname{Curl}(fF) = \nabla \times (fF)$ This is done according to chain rule.

$= \nabla f \times F + f(\nabla \times F)$

$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle \times F$

$+ \left\langle f\left(\frac{dF_3}{dy} - \frac{dF_2}{dz}, \frac{dF_1}{dz} - \frac{dF_3}{dx}, \frac{dF_2}{dx} - \frac{dF_1}{dy}\right)\right\rangle$

$= \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle \times F + f \left\langle \frac{dF_3}{dy} - \frac{dF_2}{dz}, \frac{dF_1}{dz} - \frac{dF_3}{dx}, \frac{dF_2}{dx} - \frac{dF_1}{dy} \right\rangle$

$= \nabla f \times F + f \operatorname{curl} F$

Therefore, the identity is true.

2. (8 points) Let $\mathcal{C} \subset \mathbb{R}^2$ be the part of the curve $y = x^4$ from $x = 1$ to $x = -1$. Find

$$\int_{\mathcal{C}} (1+x)\, dy - y\, dx.$$

We can define the parameterization

$r(t) = \langle t, t^4 \rangle$ from $1 \geq t \geq -1$

We can write

$F = \langle 1+x, y \rangle$.

The normal vector $\langle y'(t), -x'(t) \rangle$ is $\langle 4t^3, -1 \rangle$

Thus, we can calculate (reversed because opposite orientation)

$$-\int_{-1}^{1} \langle 1+x, y \rangle \cdot \langle 4t^3, -1 \rangle \, dt$$

$$= -2\int_0^1 4t^3 + 4t^3 x - y \; dt$$

$$= -2\int_0^1 4t^3 + 4t^4 - t^4 \, dt = -2\int_0^1 4t^3 + 3t^4 \, dt$$

$$= -2\left(\left(t^4 + \frac{3}{5}t^5\right)\Big|_0^1 \right)$$

$$= -2\left(1 + \frac{3}{5}\right)$$

$$= -2\left(\frac{8}{5}\right)$$

$$= -\frac{16}{5}.$$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle -\sin(x-z)\ ,\ \frac{y}{\sqrt{1+y^2}}\ ,\ \sin(x-z)-1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $(0,1,0)$ to $(\pi,0,\frac{\pi}{2})$.

$\frac{df}{dx} = -\sin(x-z)$ $\quad \int -\sin(x-z)\,dx = \cos(x-z)$

We can integrate and find

$$f = \cos(x-z) + g(y,z)$$

$\frac{df}{dy} = \frac{y}{\sqrt{1+y^2}}$ $\quad \int \frac{y}{\sqrt{1+y^2}}\,dy = \sqrt{1+y^2}$

We can integrate and find (using u substitution)

$$f = \cos(x-z) + \sqrt{1+y^2} + h(z)$$

$\frac{df}{dz} = \sin(x-z) - 1$ $\quad \int \sin(x-z)\,dz = \cos(x-z)$

We can integrate and find

$$f = \cos(x-z) - z$$

Combined:

$$f = \cos(x-z) + \sqrt{1+y^2} - z$$

Then $\int_C F(x,y,z) \cdot dr$

would equal $f(\pi, 0, \frac{\pi}{2}) - f(0,1,0)$

$f(\pi, 0, \frac{\pi}{2}) = \cos(\frac{\pi}{2}) + 1 - \frac{\pi}{2}$

$= 1 - \frac{\pi}{2}$

$f(0,1,0) = \cos(0) + \sqrt{2}$

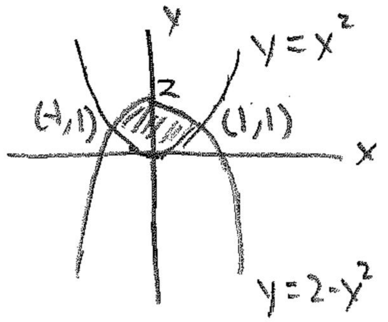
$= 1 + \sqrt{2}$

$1 - \frac{\pi}{2} - 1 - \sqrt{2}$

$\boxed{= -\frac{\pi}{2} - \sqrt{2}}$

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $y = x^2$, $y = 2 - x^2$, $z = x^2$, $z = 2 - x^2$.

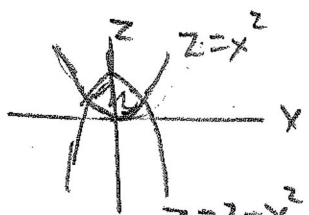
We sketch the projection



The domain D can be written as

$D = \{ -1 \leq x \leq 1,\ x^2 \leq y \leq 2 - x^2 \}$

The lower height is $z = x^2$ and the taller height is $z = 2 - x^2$.



(As seen from the projection onto x-z plane diagram)

We can set up the integral and calculate:

$$\int_{-1}^{1} \int_{x^2}^{2-x^2} 2 - x^2 - x^2 \, dy \, dx$$

$$= \int_{-1}^{1} \int_{x^2}^{2-x^2} 2 - 2x^2 \, dy \, dx$$

$$= \int_{-1}^{1} (2y - 2x^2 y) \Big|_{x^2}^{2-x^2} \, dx$$

$$= \int_{-1}^{1} 2(2 - x^2) - 2x^2(2 - x^2) - 2x^2 + 2x^4 \, dx$$

$$= \int_{-1}^{1} 4 - 2x^2 - 4x^2 + 2x^4 - 2x^2 + 2x^4 \, dy$$

$$= 2\int_{0}^{1} 4x^4 - 8x^2 + 4 \, dx$$

$$= 2\left(\left(\frac{4}{5}x^5 - \frac{8}{3}x^3 + 4x\right)\Big|_0^1\right)$$

$$= 2\left(\frac{4}{5} - \frac{8}{3} + 4\right)$$

$$= \frac{8}{5} - \frac{16}{3} + 8$$

$$= \frac{24}{15} - \frac{80}{15} + \frac{120}{15}$$

$$\boxed{= \frac{64}{15}}$$