1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

Let $F = \langle F_1, F_2, F_3 \rangle$

$$\operatorname{curl}(fF) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle \times \langle F_1, F_2, F_3 \rangle + f \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$$= \left\langle F_3 \frac{\partial f}{\partial x} - F_2 \frac{\partial f}{\partial z}, F_1 \frac{\partial f}{\partial z} - F_3 \frac{\partial f}{\partial x}, F_2 \frac{\partial f}{\partial x} - F_1 \frac{\partial f}{\partial y} \right\rangle + \left\langle f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), f\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right), f\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \right\rangle$$

2. (8 points) Let $\mathcal{C} \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find

$$\int_{\mathcal{C}} (1 + x)\, dy - y\, dx.$$

Parametrize the curve by $\vec{r}(t) = \langle t, t^5 \rangle$ for $-1 \leq t \leq 1$

Then $\vec{r}'(t) = \langle 1, 5t^4 \rangle$

So $\int_c (1+x)\,dy - y\,dx = \int_1^{-1} (t+1)5t^4\,dt - t^5\,dt$

$= \int_1^{-1} 4t^5 + 5t^4\,dt$

$= \left[\frac{4t^6}{6} + \frac{5t^5}{5}\right]_1^{-1}$

$= \left(\frac{2}{3} - 1\right) - \left(\frac{2}{3} + 1\right)$

$\boxed{\int_c (1+x)\,dy - y\,dx = -2}$

The integral $\int_c (1+x)\,dy - y\,dx$ for the curve $y = x^5$ from $x = 1$ to $x = -1$ is equal to -2.

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where $\mathcal{C}$ is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

Find a potential function f for $\vec{F}$

$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1+x^2}}$ then $\frac{1}{2}\int 2x(1+x^2)^{-\frac{1}{2}} = (1+x^2)^{\frac{1}{2}}$

So $f(x,y,z) = \sqrt{1+x^2} + g(y,z)$

Then $\frac{\partial f}{\partial y} = \cos(y-z) = \frac{\partial g}{\partial y}$

So $g(y,z) = \sin(y-z) + h(y)$

Then $\frac{\partial f}{\partial z} = -\cos(y-z)+1 = -\cos(y-z) + h'(y)$

$h'(y) = 1$

$h(y) = z$

So $f(x,y,z) = \sqrt{1+x^2} + \sin(y-z) + z$

$\nabla f = \vec{F}$ so $\vec{F}$ is conservative

Then by the Fundamental Theorem of Vector Line Integrals

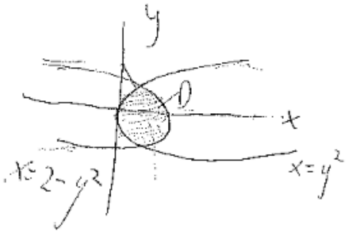
$\int_C \vec{F}(x,y,z)\cdot d\vec{r} = f(0,\pi,\frac{\pi}{2}) - f(1,0,0)$

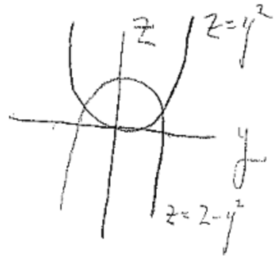
$= 1 + 1 + \frac{\pi}{2} - \sqrt{2}$

$\boxed{\int_C \vec{F}(x,y,z)\,d\vec{r} = 2 + \frac{\pi}{2} - \sqrt{2}}$

The integral $\int_C \vec{F}(x,y,z)\cdot d\vec{r}$ for C is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$ is equal to $2 + \frac{\pi}{2} - \sqrt{2}$

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.





Height $= 2 - y^2 - y^2 = 2 - 2y^2$

$D = \{-1 \le y \le 1,\ y^2 \le x \le 2 - y^2\}$

So volume of $W = \int_{-1}^{1} \int_{y^2}^{2-y^2} 2 - 2y^2 \, dx\, dy$

$= \int_{-1}^{1} \left[2x - 2xy^2\right]_{y^2}^{2-y^2}$

$= \int_{-1}^{1} \left((4 - 2y^2 - 4y^2 + 2y^4) - (2y^2 - 2y^4) \right) dy$

$= \int_{-1}^{1} 4 - 8y^2 + 4y^4 \, dy$

$= 4 \int_{-1}^{1} 1 - 2y^2 + y^4 \, dy$

$= 4 \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_{-1}^{1}$

$= 4 \left[1 - \frac{2}{3} + \frac{1}{5} - \left(-1, \frac{2}{3}, -\frac{1}{5}\right) \right]$

$= 4 \left[2 - \frac{4}{3} + \frac{2}{5} \right]$

$= 4 \left(\frac{30}{15} - \frac{20}{15} + \frac{6}{15} \right)$

$= 4 \left(\frac{16}{15} \right)$

Volume of $W = \frac{64}{15}$

The volume of region W bounded by $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$ is $\frac{64}{15}$.