Math 32B Midterm 1L

[redacted]

TOTAL POINTS

38 / 40

QUESTION 1

1 Product rule 5 / 5

✓ **+ 2 pts Correct expression for curl operator in components**

✓ **+ 1 pts Correct expression for $$\nabla f$$**

✓ **+ 1 pts Correct application of the product rule for partial derivatives**

✓ **+ 1 pts Solution clearly explained**

+ **0 pts** No credit due

QUESTION 2

2 Line integral 7 / 8

✓ **+ 4 pts Parametrized curve correctly, except possibly orientation**

+ **2 pts** Correct orientation of curve/integral

✓ **+ 2 pts Successful computation of the correct integral, except possibly orientation/sign**

✓ **+ 1 pts Clearly explaining your solution**

+ **1 pts** Bonus: sketch of curve (with or without orientation)

+ **0 pts** No points

QUESTION 3

3 Fundamental Theorem of Vector Line Integrals 12 / 12

✓ **+ 1 pts correct answer (2 - sqrt(2) + pi/2), correctly derived using potential function, and solution is clearly explained**

✓ **+ 9 pts correct potential function f(x,y,z) = sqrt(1+x^2) + sin(y-z) + z**

✓ **+ 2 pts integral is equal to f(0,pi,pi/2) - f(1,0,0)**

+ **1 pts** (incorrect) integral is equal to f(1,0,0) - f(0,pi,pi/2)

+ **0 pts** no points

+ **7 pts** partial credit for nearly correct expression for potential function

+ **3 pts** partial credit for some progress towards finding a potential function

+ **2 pts** correct expression for a parametric curve from (1,0,0) to (0,pi,pi/2) (only if no solution via potential function)

+ **2 pts** correct expression for a vector line integral using a parametric curve (only if no solution via potential function)

+ **5 pts** partial credit for incorrect integral in a solution via a parametric curve

+ **8 pts** correct answer (2 - sqrt(2) + pi/2), correctly derived via a parametric curve, and solution is clearly explained

QUESTION 4

4 Volume via a double integral 14 / 15

✓ **+ 3 pts Drawing/labelling region**

✓ **+ 1 pts solving for y limits of the domain**

✓ **+ 2 pts Correctly set-up integral (total 5 pts)**

✓ **+ 1 pts Correctly set-up integral**

✓ **+ 2 pts Correctly set-up integral**

✓ **+ 1 pts showing z = 2 - y^2 is the top surface (2 pts)**

+ **1 pts** showing z = 2 - y^2 is the top surface

✓ **+ 2 pts computation (total 3 pts)**

✓ **+ 1 pts computation**

✓ **+ 1 pts style point**

+ **0 pts** no points / blank

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Name: [redacted] DISC 3C

SID: [redacted]

Time Limit: 50 Minutes **Version** (←)

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

we can rewrite curl (fF) as $\nabla \times f F$. Computing curl(fF)

we use

$$\begin{matrix} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ fF_1 & fF_2 & fF_3 & fF_1 & fF_2 \end{matrix}$$

$$\left(\frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2)\right)\vec{i} + \left(\frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3)\right)\vec{j} + \left(\frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1)\right)\vec{k}$$

(1)

$$= \left(\frac{\partial f}{\partial y}F_3 + \frac{\partial F_3}{\partial y}f - \frac{\partial f}{\partial z}F_2 - \frac{\partial F_2}{\partial z}f\right)\vec{i} + \left(\frac{\partial f}{\partial z}F_1 + \frac{\partial F_1}{\partial z}f - \frac{\partial f}{\partial x}F_3 - \frac{\partial F_3}{\partial x}f\right)\vec{j} + \left(\frac{\partial f}{\partial x}F_2 + \frac{\partial F_2}{\partial x}f - \frac{\partial f}{\partial y}F_1 - \frac{\partial F_1}{\partial y}f\right)\vec{k}$$

(2) The curl F =

$$\begin{matrix} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{matrix} \quad \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\vec{i} + \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\vec{j} + \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\vec{k}$$

From labeled 1 equation we can collect terms and rewrite as

(3) $f\left(\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\vec{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\vec{k}\right) +$

(4) $\left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\vec{i} + \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\right)\vec{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)\vec{k}$

As we computed curl F = equation (2) so equation (3) = f curl F

and equation (4) = $\nabla f \times F$ ← as proved with equation (5). So we have shown that

curl (fF) = $\nabla f \times F$ + f curl F.

(5) $\nabla f \times F$ is computed by

$$\begin{matrix} i & j & k & i & j \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} & \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ F_1 & F_2 & F_3 & F_1 & F_2 \end{matrix} \quad \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\vec{i} + \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\right)\vec{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)\vec{k}$$

2. (8 points) Let $\mathcal{C} \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find

$$\int_{\mathcal{C}} (1+x)\,dy - y\,dx.$$

First we determine $r(t) = \langle t, t^5 \rangle$ from $-1 \le t \le 1$

Then we compute $r'(t) = \langle 1, 5t^4 \rangle$ and $N(t) = \langle 5t^4, -1 \rangle$

Finding the flux we can rewrite $\int_C F \cdot dr$ as $\int_C F(r(t)) \cdot N(t)\,dt$

Plugging into our formula, we compute (with $F = \langle 1+x, -y \rangle$)

$$\int_{-1}^{1} \langle 1+t, -t^5 \rangle \cdot \langle 5t^4, -1 \rangle\,dt$$

$$= \int_{-1}^{1} 5t^4 + 5t^5 + t^5\,dt$$

$$\int_{-1}^{1} 5t^4 + 6t^5\,dt$$

$$t^5 + t^6 \Big|_{-1}^{1} = 1+1 - (-1+1)$$

$$= 2.$$

The flux is $\int_C (1+x)\,dy - y\,dx = 2.$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z) \cdot d\mathbf{r}$ where C is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

we determine F's potential function f by:
setting $\frac{\partial f}{\partial x} = F_1$, $\frac{\partial f}{\partial y} = F_2$, $\frac{\partial f}{\partial z} = F_3$ and
integrating with the respective variables

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1+x^2}} \Rightarrow \int \frac{x}{\sqrt{1+x^2}}\,dx = \sqrt{1+x^2} + g(y,z)$$

$$\frac{\partial f}{\partial y} = \cos(y-z) \Rightarrow \int \cos(y-z)\,dy = \sin(y-z) + h(x,z)$$

$$\frac{\partial f}{\partial z} = -\cos(y-z)+1 \quad \int -\cos(y-z)+1 = \sin(y-z) + z + K(x,y)$$

By inspection we find that a $f = \sqrt{1+x^2} + \sin(y-z) + z$

using the Fundamental theorem of vector line integrals we can conclude that $\int_C F(x,y,z) \cdot dr = f(Q) - f(P)$ on the domain D

Evaluating $f(0,\pi,\frac{\pi}{2}) - f(1,0,0) =$

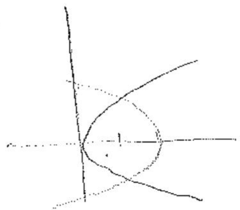
$$= \sqrt{1} + \sin(\tfrac{\pi}{2}) + \tfrac{\pi}{2} - \sqrt{2} + \sin(0) + 0$$

$$= 1 + 1 + \tfrac{\pi}{2} - \sqrt{2}$$

$$= 2 - \sqrt{2} + \tfrac{\pi}{2}$$

Thus $\int_C F \cdot dr = 2 - \sqrt{2} + \frac{\pi}{2}$.

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.



we sketch the region in x-y plane

For all $(x,y) \in D$ $2-y^2 \geq y^2$.

Finding the intersection points of $x=y^2$ and $x = 2-y^2$ we get $y^2 = 2-y^2$

$2y^2 = 2$; $y = \pm 1$.

Using this information we write our double integral as

$$\int_{-1}^{1}\int_{y^2}^{2-y^2} 2-y^2-y^2 \, dx\, dy = \int_{-1}^{1}\int_{y^2}^{2-y^2} 2-2y^2 \, dx\, dy$$

Computing the inner integral we get $\int_{y^2}^{2-y^2} 2-2y^2 \, dx\, dy =$

$$2x - 2xy^2 \Big|_{y^2}^{2-y^2} = 2(2-y^2) - 2(2-y^2)y^2 - 2y^2 + 2y^4$$

$$= 4 - 2y^2 - 4y^2 + 2y^4 - 2y^2 + 2y^4 = 4y^4 - 8y^2 + 4.$$

Subbing this back in, we compute $\int_{-1}^{1} 4y^4 - 8y^2 + 4 \, dy$

$$= \frac{4}{5}y^5 - \frac{8}{3}y^3 + 4y \Big|_{-1}^{1} = \frac{4}{5} - \frac{8}{3} + 4 + \frac{4}{5} - \frac{8}{3} + 4 =$$

$$\frac{8}{5} - \frac{16}{3} + 8 = \frac{24 - 80 + 120}{15} = \frac{64}{15}$$

Thus the volume of W is $\frac{64}{15}$ units^3.