# Math 32B Midterm 1L

[redacted]

TOTAL POINTS

## 38 / 40

QUESTION 1

### 1 Product rule 4 / 5

✓ + 2 pts Correct expression for curl operator in components

✓ + 1 pts Correct expression for $$\nabla f$$

+ 1 pts Correct application of the product rule for partial derivatives

✓ + 1 pts Solution clearly explained

+ 0 pts No credit due

Your expressions for curl(f F) and gradient of f cross product F are identical so how can you have proved the identity?

QUESTION 2

### 2 Line integral 8 / 8

✓ + 4 pts Parametrized curve correctly, except possibly orientation

✓ + 2 pts Correct orientation of curve/integral

✓ + 2 pts Successful computation of the correct integral, except possibly orientation/sign

✓ + 1 pts Clearly explaining your solution

+ 1 pts Bonus: sketch of curve (with or without orientation)

+ 0 pts No points

QUESTION 3

### 3 Fundamental Theorem of Vector Line Integrals 12 / 12

✓ + 1 pts correct answer (2 - sqrt(2) + pi/2), correctly derived using potential function, and solution is clearly explained

✓ + 9 pts correct potential function f(x,y,z) = sqrt(1+x^2) + sin(y-z) + z

✓ + 2 pts integral is equal to f(0,pi,pi/2) - f(1,0,0)

+ 1 pts (incorrect) integral is equal to f(1,0,0) - f(0,pi,pi/2)

+ 0 pts no points

+ 7 pts partial credit for nearly correct expression for potential function

+ 3 pts partial credit for some progress towards finding a potential function

+ 2 pts correct expression for a parametric curve from (1,0,0) to (0,pi,pi/2) (only if no solution via potential function)

+ 2 pts correct expression for a vector line integral using a parametric curve (only if no solution via potential function)

+ 5 pts partial credit for incorrect integral in a solution via a parametric curve

+ 8 pts correct answer (2 - sqrt(2) + pi/2), correctly derived via a parametric curve, and solution is clearly explained

QUESTION 4

### 4 Volume via a double integral 14 / 15

✓ + 3 pts Drawing/labelling region

✓ + 1 pts solving for y limits of the domain

✓ + 2 pts Correctly set-up integral (total 5 pts)

✓ + 1 pts Correctly set-up integral

✓ + 2 pts Correctly set-up integral

✓ + 1 pts showing z = 2 - y^2 is the top surface (2 pts)

+ 1 pts showing z = 2 - y^2 is the top surface

✓ + 2 pts computation (total 3 pts)

✓ + 1 pts computation

✓ + 1 pts style point

+ 0 pts no points / blank

**Math 32B - Lectures 3 & 4**
**Winter 2019**
**Midterm 1**
**2/1/2019**

**Name:** [redacted]

**SID:**

**Time Limit:** 50 Minutes

**Version** $\leftarrow$

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This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

By definition $\operatorname{curl}(f\vec{F}) = \nabla \times fF$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \left(\frac{\partial fF_3}{\partial y} - \frac{\partial fF_2}{\partial z}\right)\hat{i} - \left(\frac{\partial fF_3}{\partial x} - \frac{\partial fF_1}{\partial z}\right)\hat{j} + \left(\frac{\partial fF_2}{\partial x} - \frac{\partial fF_1}{\partial y}\right)$$

$$= \left\langle \frac{\partial}{\partial y} fF_3 - \frac{\partial}{\partial z} fF_2, \frac{\partial}{\partial z} fF_1 - \frac{\partial}{\partial x} fF_3, \frac{\partial}{\partial x} fF_2 - \frac{\partial}{\partial y} fF_1 \right\rangle.$$

We now compute $\nabla f \times F$.

$$\nabla f \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right)\hat{i} - \left(\frac{\partial f}{\partial x}F_3 - \frac{\partial f}{\partial z}F_1\right)\hat{j} + \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right)$$

$$= \left\langle \frac{\partial}{\partial y} fF_3 - \frac{\partial}{\partial z} fF_2, \frac{\partial}{\partial z} fF_1 - \frac{\partial}{\partial x} fF_3, \frac{\partial}{\partial x} fF_2 - \frac{\partial}{\partial y} fF_1 \right\rangle$$

We now compute $f \cdot \operatorname{curl} \vec{F}$.

$$f \cdot [\nabla \times F] = f \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial}{\partial y}F_3 - \frac{\partial}{\partial z}F_2\right)\hat{i} - \left(\frac{\partial}{\partial x}F_3 - \frac{\partial}{\partial z}F_1\right)\hat{j} + \left(\frac{\partial}{\partial x}F_2 - \frac{\partial}{\partial y}F_1\right)$$

$$= \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

$\therefore \operatorname{curl}(f\vec{F}) = \nabla f \times \vec{F} + f \operatorname{curl} \vec{F}$

NEXT PAGE →

We have thus proven that

$$\text{curl}(f\vec{F}) = \nabla f \times \vec{F} + f\,\text{curl}(\vec{F})$$

2. (8 points) Let $C \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find

$$\int_C (1+x)\,dy - y\,dx.$$

Since our curve is defined by $y = x^5$ for x from 1 to -1. we can re-parameterize the path C as:

$C: \vec{r}(t) = (t, t^5)$ for t from 1 to -1.

We define the curve $C'$ to be the same as C but the opposite direction, so:

$C': \vec{r}(t) = (t, t^5)$ for t from -1 to 1.

We can rewrite the given integral:

$$\int_C (1+x)\,dy - y\,dx = -\int_{C'} (1+x)\,dy - y\,dx$$

Since $\vec{r}(t) = (t, t^5)$, $dy = 5t^4$ and $dx = 1$ because $x = t$ and $y = t^5$. The integral is now:

$$= -\int_{C'} \left[(1+t)(5t^4) - (t^5)1\right] dt$$

$$= -\int_{-1}^{1} \left[5t^4 + 5t^5 - t^5\right] dt$$

$$= -\int_{-1}^{1} \left[5t^4 - 4t^5\right] dt$$

$$= -\int_{-1}^{1} 5t^4\,dt + \int_{-1}^{1} 4t^5\,dt$$

$$= -t^5\Big|_{-1}^{1} + \frac{2}{3}t^6\Big|_{-1}^{1}$$

$$= -\left[1^5 - (-1)^5\right] + \left[\frac{2}{3}(1)^6 - \frac{2}{3}(-1)^6\right]$$

$$= -(1+1) + \cancel{\left(\frac{2}{3} - \frac{2}{3}\right)}$$

$$\int_C (1+x)\,dy - y\,dx = -2$$

NEXT PAGE $\Longrightarrow$

We have thus shown that
$\int_C (1+x)\,dy - y\,dx$ for the curve defined
by $y = x^5$ from $x=1$ to $x=-1$, is
equal to $-2$.

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where $C$ is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

We wish to show that $\vec{F}$ has a potential function $f$ such that $\nabla f = \vec{F}$.
We now integrate each component of $f$, with respect to $x, y, z$, for $F_1, F_2, F_3$ of $F$, where $\vec{F} = \langle F_1, F_2, F_3 \rangle$.

1) $\int \frac{x}{\sqrt{1+x^2}}\,dx$ is the first integral we consider: we now compute the integral: $\int \frac{x}{\sqrt{1+x^2}}\,dx$.
Let $u = 1+x^2$, then $du = 2x\,dx$, and the integral can be written as $\frac{1}{2}\int \frac{2x\,dx}{\sqrt{1+x^2}}$
$= \frac{1}{2}\int \frac{du}{u^{\frac{1}{2}}} = \cancel{\frac{1}{2}}\left[\cancel{2}u^{\frac{1}{2}}\right] = \sqrt{1+x^2} + g(y,z)$

2) $\int \cos(y-z)\,dy$ is the 2nd integral we consider.
We compute: $\int \cos(y-z)\,dy$. Let $u = y-z$, $du = dy$.
$\int \cos u\,du = \sin u + C = \sin(y-z) + h(x,z)$.

3) $\int \left[-\cos(y-z)+1\right]dz$ is what we consider next.
By simple integration we determine that:
$\int\left[-\cos(y-z)+1\right]dz = \sin(y-z) + z + j(x,y)$

~~It~~ must be true that:
$\sqrt{1+x^2} + g(y,z) = \sin(y-z) + h(x,z) = \sin(y-z) + z + j(x,y)$

Therefore: $g(y,z) = z + \sin(y-z)$
$h(x,z) = \sqrt{1+x^2} + z$
$j(x,y) = \sqrt{1+x^2}$

Therefore, the potential function of $\vec{F}$, $f$, is defined as:

$$f(x,y,z) = \sqrt{1+x^2} + \sin(y-z) + z + C$$

where $C$ is a constant.

By the fundamental theorem of vector line integrals, since $f$ is a potential function of $f$, the following is true.

$$\int_C \vec{F}(x,y,z) \cdot d\vec{r} = f(P) - f(Q)$$

where $Q$ is the starting point & $P$ is the ending point. Therefore:

$$\int_C \vec{F}(x,y,z) \cdot d\vec{r} = f(0,\pi,\tfrac{\pi}{2}) - f(1,0,0)$$

$$= \left[\sqrt{1+0^2} + \sin\left(\pi - \tfrac{\pi}{2}\right) + \tfrac{\pi}{2}\right] - \left[\sqrt{1+1^2} + \sin(0-0) + 0\right]$$

$$= \left[1 + \sin\tfrac{\pi}{2} + \tfrac{\pi}{2}\right] - \left[\sqrt{2} + \sin 0\right]$$

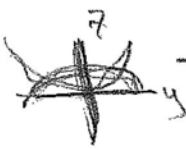
$$= \left[1 + 1 + \tfrac{\pi}{2}\right] - \left[\sqrt{2}\right]$$

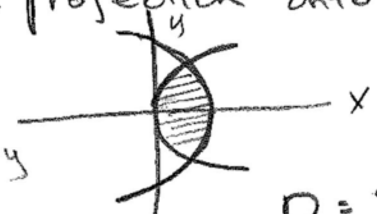
$$= 2 + \tfrac{\pi}{2} - \sqrt{2}$$

We have proven that $\int_C \vec{F}(x,y,z) \cdot d\vec{r} = 2 + \frac{\pi}{2} - \sqrt{2}$ for any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$ where $\vec{F} = \left\langle \frac{x}{\sqrt{1+x^2}}, \cos(y-z), -\cos(y-z)+1 \right\rangle$.

4. (15 points) ~~Find the~~ volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.

The Projection onto the xy-plane is as follows:




The projection can be written as a horizontally simple domain:

$D = \{-1 \le y \le 1,\ y^2 \le x \le 2 - y^2\}$.

The volume between two surfaces on a domain D is defined as:

$$V = \iint_D (g_{hi}(x,y) - g_{lo}(x,y))\, dA = \int_c^d \int_a^b (g_{hi} - g_{lo})\, dx\, dy.$$

$[z = 2 - y^2] > [z = y^2]$ for all $x, y, z$ in $D$, so the integral can be written as:

$$\int_{-1}^{1} \int_{y^2}^{2-y^2} (2 - y^2 - y^2)\, dx\, dy. = \int_{-1}^{1} \int_{y^2}^{2-y^2} (2 - 2y^2)\, dx\, dy.$$

We compute that integral:

$$= \int_{-1}^{1} (2 - 2y^2) \int_{y^2}^{2-y^2} dx\, dy$$

$$= \int_{-1}^{1} (2 - 2y^2)\ x \Big|_{y^2}^{2-y^2} dy$$

$$= \int_{-1}^{1} (2 - 2y^2) [2 - y^2 - y^2]\, dy$$

$$= \int_{-1}^{1} (2 - 2y^2)(2 - 2y^2)\, dy$$

$$= \int_{-1}^{1} (4 - 8y^2 + 4y^4)\, dy$$

$$= 4y - \frac{8}{3}y^3 + \frac{4}{5}y^5 \Big|_{-1}^{1}$$

$$= \left[4 - \frac{8}{3}(1) + \frac{4}{5}(1)\right] - \left[4(-1) - \frac{8}{3}(-1)^3 + \frac{4}{5}(-1)^5\right]$$

$$= \left[4 - \frac{8}{3} + \frac{4}{5}\right] - \left[-4 + \frac{8}{3} - \frac{4}{5}\right]$$

$$= 4 - \frac{8}{3} + \frac{4}{5} + 4 - \frac{8}{3} + \frac{4}{5} = 8 - \frac{16}{3} + \frac{8}{5} = \frac{120}{15} - \frac{80}{15} + \frac{24}{15}$$

$\begin{array}{r} 4 \\ 15 \\ \times 8 \\ \hline 120 \end{array}$ $\begin{array}{r} 16 \\ \times 5 \\ \hline 80 \end{array}$

$$\frac{120}{15} - \frac{80}{15} + \frac{24}{15} = \frac{40}{15} + \frac{24}{15} = \frac{64}{15}$$

We have shown that the volume of the region $W$ bounded by $x = y^2$, $x = 2 - y^2$, $z = y^2$, and $z = 2 - y^2$ has a volume of:

$$\frac{64}{15}.$$