1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

$$\operatorname{curl}(fF) = \nabla \times (fF) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix}$$

$$\left(\frac{\partial fF_3}{\partial y} - \frac{\partial fF_2}{\partial z}\right) i - \left(\frac{\partial fF_3}{\partial x} - \frac{\partial fF_1}{\partial z}\right) j + \left(\frac{\partial fF_2}{\partial x} - \frac{\partial fF_1}{\partial y}\right) k$$

By the product rule, this is

$$\left(\boxed{\frac{\partial f}{\partial y} F_3} + \frac{\partial F_3}{\partial y} f \boxed{- \frac{\partial f}{\partial z} F_2} - \frac{\partial F_2}{\partial z} f\right) i + \left(\boxed{-\frac{\partial f}{\partial x} F_3} - \frac{\partial F_3}{\partial x} f + \boxed{\frac{\partial f}{\partial z} F_1} + \frac{\partial F_1}{\partial z} f\right) j$$
$$+ \left(\boxed{\frac{\partial f}{\partial x} F_2} + \frac{\partial F_2}{\partial x} f \boxed{- \frac{\partial f}{\partial y} F_1} - \frac{\partial F_1}{\partial y} f\right) k$$

$$\boxed{\nabla f \times F} = \begin{vmatrix} i & j & k \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \quad \left(\frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2\right) i + \left(-\frac{\partial f}{\partial x} F_3 + \frac{\partial f}{\partial z} F_1\right) j$$
$$+ \left(\frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial z} F_1\right) k$$

$$f \operatorname{curl} F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} f - \frac{\partial F_2}{\partial z} f\right) + \left(-\frac{\partial F_3}{\partial x} f + \frac{\partial F_1}{\partial z} f\right) j$$
$$+ \left(\frac{\partial F_2}{\partial x} f - \frac{\partial F_1}{\partial y} f\right) k$$

Summing these two, you get $\operatorname{curl}(fF) = \nabla f \times F + f \operatorname{curl} F$.
I have boxed the terms of $\nabla f \times F$, and circled the terms of $f \operatorname{curl} F$.

2. (8 points) Let $\mathcal{C} \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find

$$\int_{\mathcal{C}} (1+x)\,dy - y\,dx.$$

The curve can be paramaterized as $r(t) = \langle t, t^5 \rangle$ for $1 \geq t \geq -1$

So, $r'(t) = \langle 1, 5t^4 \rangle$

$$\int_C (1+x)\,dy - y\,dx = \int_1^{-1} (1+t)5t^4 - t^5 \, dt$$

$$= \int_1^{-1} 5t^4 + 5t^5 - t^5 \, dt = \int_1^{-1} 5t^4 + 4t^5 \, dt$$

$$= t^5 + \frac{2}{3}t^6 \Big|_1^{-1}$$

$$\left(-1 + \frac{2}{3}\right) - \left(1 + \frac{2}{3}\right)$$

$$= -\frac{1}{3} - \frac{5}{3} = \boxed{-2}$$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

we assume F is conservative,

$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1+x^2}}$, so integrating w.r.t x; $f(x,y,z) =$

$\int \frac{x}{\sqrt{1+x^2}}\,dx$, Let $u = 1+x^2$, $du = 2x\,dx$ $\int \frac{x}{\sqrt{1+x^2}}\,dx = \frac{1}{2}\int u^{-1/2}\,du$

$= \frac{1}{2}(2u^{1/2}) = \sqrt{1+x^2} + g(y,z)$

$\frac{\partial f}{\partial y} = \cos(y-z)$, so $f(x,y,z) = \int \cos(y-z)\,dy$

$= \sin(y-z) + g(z)$

$\frac{\partial f}{\partial z} = -\cos(y-z)+1$, so $f(x,y,z) = \int -\cos(y-z)+1\,dz$

$= \sin(y-z) + z$

so, $f(x,y,z) = \sqrt{1+x^2} + \sin(y-z) + z$

Since f is conservative, $\int_C F(x,y,z) = f(Q) - f(P)$, so

$\int_C F(x,y,z) = f(0,\pi,\pi/2) - f(1,0,0)$

$= 2 + \pi/2 - \sqrt{2}$

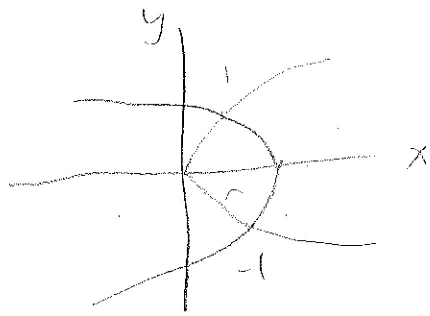
4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.

$2 - y^2 = y^2$

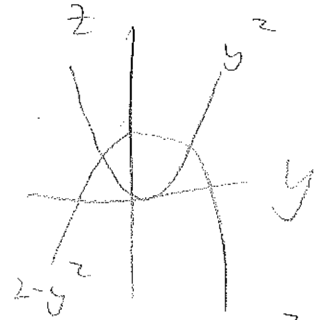
$2y^2 = 2$

$y^2 = 1$

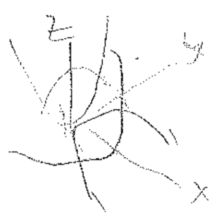
On the xy plane



On the y-z plane



The base of the solid we are interested in is given by $\{(x,y) \mid -1 \le y \le 1,\ y^2 \le x \le 2 - y^2\}$



we can see that the height ~~for~~ ~~...~~ at some point x, y is given by $2 - y^2 - y^2$

So, the volume is given by

$$\int_{-1}^{1} \int_{y^2}^{2-y^2} 2 - 2y^2 \, dx\, dy$$

$$= \int_{-1}^{1} 2x - 2xy^2 \Big|_{x=y^2}^{2-y^2} dy = \int_{-1}^{1} 2x(1-y^2) \Big|_{x=y^2}^{2-y^2} dy$$

$$= \int_{-1}^{1} 2(2-y^2)(1-y^2) - 2y^2(1-y^2) \, dy$$

$$= \int_{-1}^{1} 2(y^4 - 3y^2 + 2) - 2y^2 + 2y^4 \, dy$$

(back)

$= \int_{-1}^{1} 4y^4 - 8y^2 + 4 \, dy$

$= \frac{4y^5}{5} - \frac{8y^3}{3} + 4y \Big|_{-1}^{1}$

$= \frac{4}{5} - \frac{8}{3} + 4 - \left(\frac{-4}{5} + \frac{8}{3} - 4\right)$

$= \frac{8}{5} - \frac{16}{3} + 8 = \frac{24 - 80 + 120}{15}$

$\boxed{= \frac{64}{15}}$

$\begin{array}{r} 48 \\ 16 \\ \hline 64 \end{array}$

$2x - 2xy^2 \Big|_{y^2}^{2-y^2}$

$2x(1-y^2)$

$(4-2y^2)(1-y^2) - 2y^2(1-y^2)$

$2y^4 - 6y^2 + 4 - 2y^2 + 2y^4$

$= 4y^4 - 8y^2 + 4$