Math 32B Midterm 1L

[redacted]

TOTAL POINTS

38 / 40

QUESTION 1

1 Product rule 5 / 5

- ✓ + **2 pts** **Correct expression for curl operator in components**
- ✓ + **1 pts** **Correct expression for $$\nabla f$$**
- ✓ + **1 pts** **Correct application of the product rule for partial derivatives**
- ✓ + **1 pts** **Solution clearly explained**
- + **0 pts** No credit due

Surely you mean product rule, not chain rule?

QUESTION 2

2 Line integral 8 / 8

- ✓ + **4 pts** **Parametrized curve correctly, except possibly orientation**
- ✓ + **2 pts** **Correct orientation of curve/integral**
- ✓ + **2 pts** **Successful computation of the correct integral, except possibly orientation/sign**
- ✓ + **1 pts** **Clearly explaining your solution**
- ✓ + **1 pts** **Bonus: sketch of curve (with or without orientation)**
- + **0 pts** No points

QUESTION 3

3 Fundamental Theorem of Vector Line Integrals 12 / 12

- ✓ + **1 pts** **correct answer (2 - sqrt(2) + pi/2), correctly derived using potential function, and solution is clearly explained**
- ✓ + **9 pts** **correct potential function f(x,y,z) = sqrt(1+x^2) + sin(y-z) + z**
- ✓ + **2 pts** **integral is equal to f(0,pi,pi/2) - f(1,0,0)**
- + **1 pts** (incorrect) integral is equal to f(1,0,0) - f(0,pi,pi/2)
- + **0 pts** no points
- + **7 pts** partial credit for nearly correct expression for potential function
- + **3 pts** partial credit for some progress towards finding a potential function
- + **2 pts** correct expression for a parametric curve from (1,0,0) to (0,pi,pi/2) (only if no solution via potential function)
- + **2 pts** correct expression for a vector line integral using a parametric curve (only if no solution via potential function)
- + **5 pts** partial credit for incorrect integral in a solution via a parametric curve
- + **8 pts** correct answer (2 - sqrt(2) + pi/2), correctly derived via a parametric curve, and solution is clearly explained

QUESTION 4

4 Volume via a double integral 13 / 15

- ✓ + **3 pts** **Drawing/labelling region**
- + **1 pts** solving for y limits of the domain
- ✓ + **2 pts** **Correctly set-up integral (total 5 pts)**
- ✓ + **1 pts** **Correctly set-up integral**
- ✓ + **2 pts** **Correctly set-up integral**
- ✓ + **1 pts** **showing z = 2 - y^2 is the top surface (2 pts)**
- + **1 pts** showing z = 2 - y^2 is the top surface
- ✓ + **2 pts** **computation (total 3 pts)**
- ✓ + **1 pts** **computation**
- ✓ + **1 pts** **style point**
- + **0 pts** no points / blank

Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Name: [redacted]

[redacted]

Time Limit: 50 Minutes

Version ($\leftarrow$)

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

We know that $\operatorname{curl}(\vec{F}) = \nabla \times \vec{F}$ so, $\operatorname{curl}(f\vec{F}) = \nabla \times f(\vec{F}) = \nabla \times \langle fF_1, fF_2, fF_3 \rangle$

$\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle$, so we take the cross product of ∇ and $f(\vec{F})$ to obtain:

$$\operatorname{curl}(f\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ fF_1 & fF_2 & fF_3 \end{vmatrix} = \left(\frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z} \right)\hat{i} - \left(\frac{\partial(fF_3)}{\partial x} - \frac{\partial(fF_1)}{\partial z} \right)\hat{j} + \left(\frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y} \right)\hat{k}$$

$$= \left\langle \frac{\partial(fF_3)}{\partial y} - \frac{\partial(fF_2)}{\partial z}, \frac{\partial(fF_1)}{\partial z} - \frac{\partial(fF_3)}{\partial x}, \frac{\partial(fF_2)}{\partial x} - \frac{\partial(fF_1)}{\partial y} \right\rangle$$

Now, to be able to compute these partial derivatives using the chain rule, which should be able to complete our proof,

$$= \left\langle \left(\frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\right) + f\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right), \left(\frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_2\right) + f\left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial z}\right), \left(\frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1\right) + f\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x}\right) \right\rangle$$

We can split this vector into two vectors, which should be equal to $\nabla f \times \vec{F} + f \operatorname{curl} \vec{F}$,

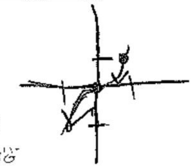
$$= \left\langle \frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2, \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3, \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1 \right\rangle + f\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle$$

We then notice that the first vector is equal to $\nabla f \times \vec{F}$ as partials are taken of each f and the cross product is taken with the vector $\vec{F}$. As well the second vector is just f multiplied by the curl of $\vec{F}$ as we know that $\operatorname{curl} \vec{F} = \langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \rangle$, so we can write the above line as:

$$= \underbrace{\nabla f \times \vec{F}}_{\text{gradient of } f \text{ cross } \vec{F}} + f \operatorname{curl} \vec{F}$$, which is what we wanted to show.

Therefore, we were able to successfully show that the $\operatorname{curl}(f\vec{F}) = \nabla f \times \vec{F} + f \operatorname{curl} \vec{F}$ through taking the cross product and applying the chain rule.

∗ 2. (8 points) Let $C \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find



$$\int_C (1+x)\,dy - y\,dx.$$

vector line integral b/c dot product produces this, which is why we can replace it

we are going in the opposite direction than we'd expect, so order actually matters.

First, we need to parametrize the curve such that if $x=t$, then $y=t^5$, so $\vec{r}(t) = \langle x(t), y(t)\rangle = \langle t, t^5\rangle$, [for $-1 \le t \le 1$] We also see from the problem that we need to factor out a dt from the double integral to be able to have this integral in a form that we can compute.

$$\int_C (1+x)\,dy - y\,dx = \int_C \left[(1+x)\frac{dy}{dt} - y\frac{dx}{dt}\right]dt.$$ Now, we need to find $\frac{dx}{dt}$ and $\frac{dy}{dt}$

We know $x(t) = t$, so $\frac{dx}{dt} = x'(t) = 1$ and $y(t) = t^5$, so $\frac{dy}{dt} = y'(t) = 5t^4$. Now, we substitute $x(t)=t$ for x, $y(t)=t^5$ for y, $\frac{dx}{dt}=1$ for $\frac{dx}{dt}$ and $\frac{dy}{dt} = 5t^4$ for $\frac{dy}{dt}$ into the integral.

$$\int_C (1+x)\,dy - y\,dx = \int_1^{-1} \left((1+t)5t^4 - t^5(1)\right)dt = -\int_{-1}^{1}(5t^4 + 5t^5 - t^5)\,dt = -\int_{-1}^{1}(5t^4 - 4t^5)\,dt$$

$$= -\left(t^5 - \frac{4}{6}t^6\right)\Big]_{-1}^{1} = -\left(1-\frac{4}{6}\right) + \left(-1-\frac{4}{6}\right) = -1+\frac{4}{6}-1-\frac{4}{6} = -2.$$

Therefore, the result of the line integral is -2 because the direction specified in the problem was from $x=1$ to $x=-1$ instead of the other way around. Because an integral in the form above is a result of the dot product of $\vec{F}(x,y)$ with $\vec{r}'(t)$, the direction stated does matter and must be taken into account!

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $\overset{P}{(1,0,0)}$ to $\overset{Q}{(0,\pi,\frac{\pi}{2})}$.

Since the problem is asking for any smooth curve, it shows that we should be looking for a potential function of $\vec{F}$, so that we can apply the Fundamental Theorem of Vector Line Integrals to be able to compute the vector line integral.

If a potential function f exists, such that $\nabla f = \vec{F}$, then the partial integrals of all the components of $\vec{F}$ should be equal. So, we compute all three partial integrals; where $\vec{F} = \langle F_1, F_2, F_3 \rangle$

- $f(x,y,z) = \int F_1\,dx = \int \frac{x}{\sqrt{1+x^2}}\,dx$ let $u = 1+x^2$, $du = 2x\,dx$, substitute to get $\frac{1}{2}\int u^{-\frac{1}{2}}du = \frac{1}{2}\left(\frac{2}{1}\right)u^{\frac{1}{2}} + f(y,z)$

Now we subsitute back in for u to obtain the value of this partial integral: $= \sqrt{1+x^2} + f(y,z)$

- We compute: $f(x,y,z) = \int F_2\,dy = \int \cos(y-z)\,dy = \sin(y-z) + g(x,z)$
- Finally, we compute the third partial integral such that let $u = y-z$, $du = -dz$

$$f(x,y,z) = \int F_3\,dz = \int(-\cos(y-z)+1)\,dz = \sin(y-z) + z + h(x,y)$$

If we are able to set these three partial integrals equal to each other, we will have obtained a potential function f for $\vec{F}$.

$$\sqrt{1+x^2} + f(y,z) = \sin(y-z) + g(x,z) = \sin(y-z) + z + h(x,y).$$

These partial integrals are equal for $f(y,z) = \sin(y-z)+z$, $g(x,z) = \sqrt{1+x^2}+z$, and $h(x,y) = \sqrt{1+x^2}$.
This allows us to obtain the potential function:

$$f(x,y,z) = \sqrt{1+x^2} + \sin(y-z) + z + C \quad \text{for any constant } C.$$

Since we were able to obtain a potential function f such that $\nabla f = \vec{F}$, we are able to say that $\vec{F}$ is conservative. This allows us to apply the Fundamental Theorem of Vector Line Integrals to be able to compute $\int_C \vec{F}(x,y,z)\cdot d\vec{r} = f(Q) - f(P)$, for our potential function above.

$$= f(0,\pi,\tfrac{\pi}{2}) - f(1,0,0) = \sqrt{1+0} + \sin\tfrac{\pi}{2} + \tfrac{\pi}{2} + C - (\sqrt{2} + \sin 0 + 0 + C)$$

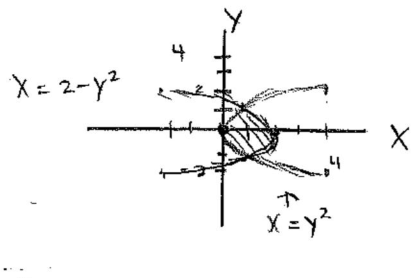
$$= 1 + 1 + \tfrac{\pi}{2} + C - \sqrt{2} - C = \boxed{2 - \sqrt{2} + \tfrac{\pi}{2}}$$

Therefore, by applying the fundamental theorem of vector line integrals, we were able to compute the vector line integral described in the problem.

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.

Start by drawing the projection of the surfaces onto the XY plane, so that we know what domain to integrate over.

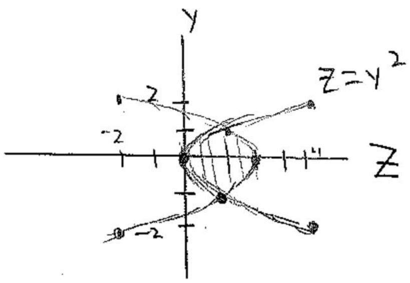
XY projection:



The domain is sandwiched between the two horizontally simple regions $x = 2 - y^2$ and $x = y^2$. We can write the domain of this region as $D = \{-1 \le y \le 1,\ y^2 \le x \le 2 - y^2\}$

Now, we need to find the height over this domain to be able to find the volume. To do this, we can look at the projection of the surfaces on the YZ plane.

YZ projection.



The height of the volume is determined by the region sandwiched between the functions $Z = 2 - y^2$ and $Z = y^2$, which allows us to write that the height is $f(x,y) = 2 - 2y^2$.

To find the volume of the region W bounded by the surfaces above, we must compute the double integral of the height over our domain. We write $\text{Volume}(W) = \iint_D f(x,y)\,dA = \int_{y=-1}^{1}\int_{x=y^2}^{2-y^2} (2-2y^2)\,dx\,dy$

We first compute the inner integral by applying Fubini's theorem:

$$\int_{y=-1}^{1} (2x - 2xy^2)\Big]_{x=y^2}^{2-y^2} dy = \int_{y=-1}^{1} 2(2-y^2) - 2(2-y^2)y^2 - [2y^2 - 2y^2y^2]\,dy$$

Now, we compute the outer integral,

$$= \int_{y=-1}^{1} (4 - 2y^2 - 4y^2 + 2y^4 - 2y^2 + 2y^4)\,dy = \int_{y=-1}^{1} (4 - 8y^2 + 4y^4)\,dy = 4y - \frac{8}{3}y^3 + \frac{4}{5}y^5\Big]_{-1}^{1}$$ and evaluate to

get $4(1) - \frac{8}{3}(1)^3 + \frac{4}{5}(1)^5 - [4(-1) - \frac{8}{3}(-1) + \frac{4}{5}(-1)] = 8 - \frac{16}{3} + \frac{8}{5} = \frac{8(15)}{15} - \frac{16(5)}{15} + \frac{8(3)}{15}$

$$= \frac{120}{15} - \frac{80}{15} + \frac{24}{15} = \frac{40}{15} + \frac{24}{15} = \boxed{\frac{64}{15} \text{ units}^3}$$

Therefore, by computing the double integral of the height, $f(x,y)$ over the domain D, we obtain the volume of the region W.