Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

Name: ______________________

SID: ______________________

Time Limit: 50 Minutes

Version $\leftarrow$

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x, y, z)$ be a smooth scalar field and $\mathbf{F}(x, y, z)$ be a smooth vector field. Show that

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F}.$$

Solution: Let $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$. Using the definition of curl we may write

$$\begin{aligned}
\operatorname{curl}(f\mathbf{F}) &= \left\langle \frac{\partial}{\partial y}(fF_3) - \frac{\partial}{\partial z}(fF_2)\,,\ \frac{\partial}{\partial z}(fF_1) - \frac{\partial}{\partial x}(fF_3)\,,\ \frac{\partial}{\partial x}(fF_2) - \frac{\partial}{\partial y}(fF_1) \right\rangle \\
&= \left\langle \frac{\partial f}{\partial y}F_3 - \frac{\partial f}{\partial z}F_2\,,\ \frac{\partial f}{\partial z}F_1 - \frac{\partial f}{\partial x}F_3\,,\ \frac{\partial f}{\partial x}F_2 - \frac{\partial f}{\partial y}F_1 \right\rangle \\
&\quad + f\left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\,,\ \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\,,\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \\
&= \nabla f \times \mathbf{F} + f \operatorname{curl} \mathbf{F},
\end{aligned}$$

where the second equality follows from the product rule.

2. (8 points) Let $\mathcal{C} \subset \mathbb{R}^2$ be the part of the curve $y = x^5$ from $x = 1$ to $x = -1$. Find

$$\int_{\mathcal{C}} (1 + x)\, dy - y\, dx.$$

Solution: We parameterize the curve $\mathcal{C}$ using $\mathbf{r}(t) = \langle -t, -t^5 \rangle$ for $-1 \leq t \leq 1$. We then have $dx = -dt$ and $dy = -5t^4\, dt$ so

$$\begin{aligned}
\int_{\mathcal{C}} (1 + x)\, dy - y\, dx &= \int_{-1}^{1} (1 - t)(-5t^4) - (-t^5)(-1)\, dt \\
&= \int_{-1}^{1} -5t^4 + 4t^5\, dt \\
&= \left[-t^5 + \frac{2}{3} t^6 \right]_{t=-1}^{t=1} \\
&= -2.
\end{aligned}$$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle .$$

Find $\int_{\mathcal{C}} \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where $\mathcal{C}$ is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

Solution: We first show that $\mathbf{F}$ is conservative by finding a potential function f for $\mathbf{F}$. Indeed, if f exists then by comparing the first components

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1+x^2}},$$

so $f(x,y,z) = \sqrt{1+x^2} + g(y,z)$. By comparing second components we see that

$$\frac{\partial g}{\partial y} = \cos(y-z),$$

so $g(y,z) = \sin(y-z) + h(z)$. By comparing third components we see that

$$\frac{dh}{dz} = 1,$$

so $h(z) = z$ will do. Thus

$$f(x,y,z) = \sqrt{1+x^2} + \sin(y-z) + z,$$

is a potential function for $\mathbf{F}$.

By the Fundamental Theorem of Vector Line Integrals we then have

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = f(0,\pi,\frac{\pi}{2}) - f(1,0,0) = 2 + \frac{\pi}{2} - \sqrt{2}.$$

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$.

Solution: Let $\mathcal{D} = \{-1 \leq y \leq 1\ ,\ y^2 \leq x \leq 2 - y^2\}$. Then we may write $\mathcal{W} = \{(x, y) \in \mathcal{D}\ ,\ y^2 \leq z \leq 2 - y^2\}$. In particular, applying Fubini's Theorem, we may compute the volume of $\mathcal{W}$ to be

$$\begin{aligned}
\text{Volume}(\mathcal{W}) &= \iint_{\mathcal{D}} 2(1 - y^2)\, dA \\
&= \int_{-1}^{1} \int_{y^2}^{2-y^2} 2(1 - y^2)\, dx dy \\
&= \int_{-1}^{1} 4(1 - y^2)^2\, dy \\
&= 8 \int_{0}^{1} 1 - 2y^2 + y^4\, dy \\
&= 8 \left(1 - \frac{2}{3} + \frac{1}{5}\right) \\
&= \frac{64}{15}
\end{aligned}$$