Math 32B - Lectures 3 & 4
Winter 2019
Midterm 1
2/1/2019

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Time Limit: 50 Minutes

Version ($\leftarrow$)

This exam contains 10 pages (including this cover page) and 4 problems. There are a total of 40 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.
- At least one point on each problem will be for clearly explaining your solution, as on the homeworks.

1. (5 points) Let $f(x,y,z)$ be a smooth scalar field and $\mathbf{F}(x,y,z)$ be a smooth vector field. Show that

scalar zero F_1 etc needed

$$\operatorname{curl}(f\mathbf{F}) = \nabla f \times \mathbf{F} + f \operatorname{curl}\mathbf{F}.$$

Set $u = fF$

curl (fF)

By def of curl $\operatorname{curl}\vec{u} =$

$$\begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} = \left\langle \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}, -\frac{\partial u_3}{\partial x} + \frac{\partial u_1}{\partial z}, \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right\rangle$$

by substitution

We compute use the product rule

$$\operatorname{curl}(fF) = \Big\langle \frac{\partial f}{\partial y} F_3 + \underline{f \frac{\partial F_3}{\partial y}} - \frac{\partial f}{\partial z} F_2 - \underline{f \frac{\partial F_2}{\partial z}},$$
$$- \frac{\partial f}{\partial x} F_3 - \underline{f \frac{\partial F_3}{\partial x}} + \frac{\partial f}{\partial z} F_1 + \underline{f \frac{\partial F_1}{\partial z}},$$
$$+ \frac{\partial f}{\partial x} F_2 + \underline{f \frac{\partial F_2}{\partial x}} - \frac{\partial f}{\partial y} F_1 - \underline{f \frac{\partial F_1}{\partial y}} \Big\rangle$$

2nd and 4th terms give

$f \operatorname{curl} F$

1st and 3rd terms give

$\nabla f \times F$

By substitution

$$\boxed{\operatorname{curl}(fF) = \nabla f \times F + f \operatorname{curl} F}$$

What was shown above

$$\nabla f \times F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial f}{\partial y} F_3 - \frac{\partial f}{\partial z} F_2, -\frac{\partial f}{\partial x} F_3 + \frac{\partial f}{\partial z} F_1, \frac{\partial f}{\partial x} F_2 - \frac{\partial f}{\partial y} F_1 \right\rangle$$

By inspection

$f \operatorname{curl} F = f$ (formula for curl $\vec{F}$) (same as above but replace u with F)

$N\,dx + M\,dy$ $\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ $t = x$

from $(-1,1)$

2. (8 points) Let $C \subset \mathbb{R}^2$ be the part of the curve $\boxed{y = x^5}$ from $x = 1$ to $x = -1$. Find

$y = t^5$

Vector Line Integral $\int_C (1+x)\,dy - y\,dx.$

$\langle -y, 1+x \rangle \cdot \langle dx, dy \rangle$

By definition of vector line integral $\int_C \vec{F}(\vec{r}(t)) \cdot r'(t)$ $r'(t)$

parameterize $\langle -y, 1+x \rangle \cdot \langle dx, dy \rangle$

$r(t) = \langle t, t^5 \rangle$

$r'(t) = \langle 1, 5t^4 \rangle$

$\vec{F}(\vec{r}(t)) = \langle -t^5, 1+t \rangle$

We compute $\int_{-1}^{1} \langle -t^5, 1+t \rangle \cdot \langle 1, 5t^4 \rangle \, dt$

$\int_{-1}^{1} -t^5 + 5t^4 + 5t^5 \, dt$

$\int_{+1}^{-1} 4t^5 + 5t^4 \, dt$

$\frac{2}{3} t^6 + t^5 \Big|_{+1}^{-1}$

We compute $\frac{2}{3}(1) - 1 - \left(\frac{2}{3} + 1\right)$

$\frac{2}{3} - \frac{2}{3} - 1 - 1 = \boxed{-2}$

$y = x^5$ $x = t$

$\sqrt{1 + (5x^4)^2}$

$y = x^5$ $\langle t, t^5 \rangle$

$\int_C -y\,dx + (1+x)\,dy$
$F(\vec{r}(t)) \cdot r'(t)$

$\langle t, t^5 \rangle$

$\int \langle -t^5, 1+t \rangle \cdot \langle 1, 5t^4 \rangle$

$\int_C -t^5 + 5t^4 + 5t^5 \; dt$

$\int_C 4t^5 + 5t^4 \; dt$

$\frac{2}{3} t^6 + t^5 \Big|_{1}^{-1}$

$\frac{2}{3}(1) - 1 - \frac{2}{3} - 1 = \boxed{-2}$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,d$

3. (12 points) Let

$$\mathbf{F}(x,y,z) = \left\langle \frac{x}{\sqrt{1+x^2}}\ ,\ \cos(y-z)\ ,\ -\cos(y-z)+1 \right\rangle.$$

Find $\int_C \mathbf{F}(x,y,z)\cdot d\mathbf{r}$ where C is any smooth curve from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$.

any smooth curve → need to find a potential function

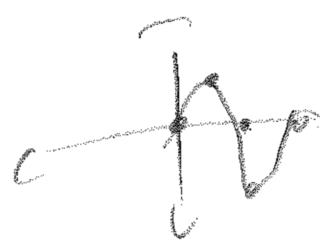
$\int \frac{\partial f}{\partial x}$ $\int \frac{x}{\sqrt{1+x^2}}\,dx$ $\quad \frac{1}{2}u^{-1/2}$ $\quad u^{1/2}$

$f = (1+x^2)^{1/2} + g(y,z)$

$\frac{\partial f}{\partial y} = \int \frac{\partial g(y,z)}{\partial y}$ $\int \cos(y-z)$

We compute

$g(y,z) = h(z) + \sin(y-z)$



$\frac{\partial f}{\partial z} = (1+x^2)^{1/2} + \sin(y-z) + h(z)$

$= -\cos(y-z) + \frac{\partial h}{\partial z} = -\cos(y-z)+1$

$\int \frac{\partial h}{\partial z} = \int 1$ $\quad h = z$

$f = (1+x^2)^{1/2} + \sin(y-z) + z$ For potential

By FTOVLI evaluate from $(1,0,0)$ to $(0,\pi,\frac{\pi}{2})$

$= f(0,\pi,\frac{\pi}{2}) - f(1,0,0)$

$(1)^{1/2} + \sin(\frac{\pi}{2}) + \frac{\pi}{2} - (\sqrt{2} + \sin 0 + 0)$

$1 + 1 + \frac{\pi}{2} - \sqrt{2} = \boxed{2 + \frac{\pi}{2} - \sqrt{2}} = \frac{4+\pi-2\sqrt{2}}{2}$

Check

$P(0, \pi, \frac{\pi}{2}) - P(1, 0, 0)$

$F = (1+x^2)^{1/2} + \sin(y-z) + z$

$\frac{\partial f}{\partial x} = \frac{x}{(1+x^2)^{1/2}}$

$\frac{\partial f}{\partial y} = \cos(y-z)$

$\frac{\partial f}{\partial z} \quad -\cos(y-z) + 1$

$(1+0)^{1/2} + \sin(\frac{\pi}{2}) + \frac{\pi}{2} - (\sqrt{2} + \sin 0 + 0)$

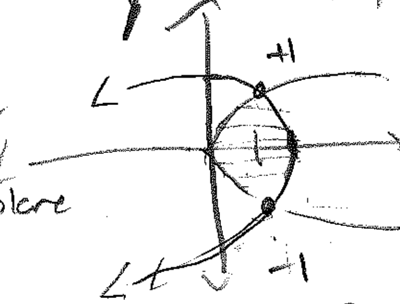
$1 + 1 + \frac{\pi}{2} - \sqrt{2}$

$2 + \frac{\pi}{2} - \sqrt{2}$

4. (15 points) Find the volume of the region $\mathcal{W}$ bounded by the surfaces $x = y^2$, $x = 2 - y^2$, $z = y^2$, $z = 2 - y^2$. ← top region

$\boxed{\text{Height} = 2 - y^2 - y^2}$

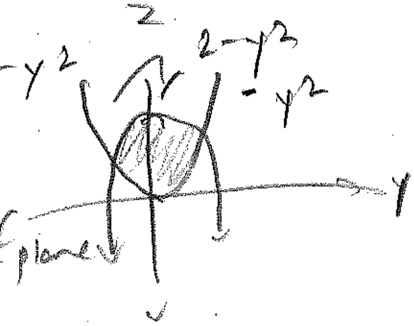
Draw a picture



xy plane

$y^2 = 2 - y^2$

$y = \pm 1$



yz plane

By Fubini's Theorem

$$\int_{-1}^{1}\int_{y^2}^{2-y^2} 2-2y^2 \, dx \, dy$$

$$\int_{-1}^{1} x(2-2y^2)\Big|_{y^2}^{2-y^2} dy \qquad -2(y^2-1)$$

$$\int_{-1}^{1} (2-2y^2)(2-2y^2) \, dy$$

We compute

$$4\int_{-1}^{1} (y^2-1)^2 \, dy$$

$$4\int_{-1}^{1} y^4 - 2y^2 + 1$$

$$4\left(\frac{y^5}{5} - \frac{2}{3}y^3 + y \Big|_{-1}^{1}\right)$$

$$\frac{1}{5} - \frac{2}{3} + 1 - 1\left(-\frac{1}{5} + \frac{2}{3} - 1\right)$$

$$4\left(\frac{2}{5} - \frac{4}{3} + 2\right)$$

$\frac{16}{64}$

$\frac{6}{15} - \frac{20}{15} + \frac{30}{15}$

$4\left(\frac{16}{15}\right)$

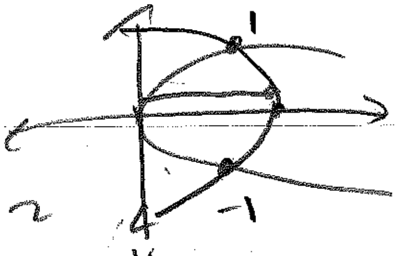
$\frac{1}{5} - \frac{2}{3} + 1$

$\boxed{\frac{64}{15}}$

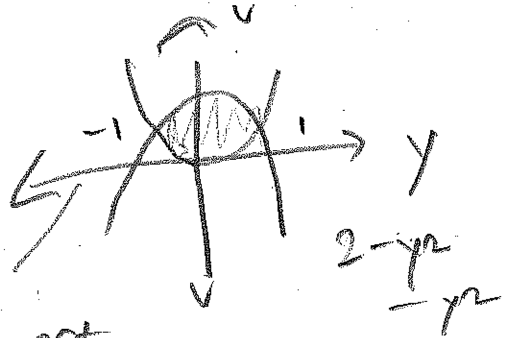
$\frac{6}{15} - \frac{20}{15} + \frac{30}{15}$

Check

$y \quad x=y^2 \quad x=2-y^2 \quad 2-y^2 \quad z=2-y^2$



$$\int_{-1}^{1}\int_{y^2}^{2-y^2} 2-2y^2 \, dx\, dy$$



correct bound (same as one)

$$\int_{-1}^{1} x(2-2y^2)\Big|_{y^2}^{2-y^2}$$

$$\int_{-1}^{1} (2-2y^2)(2-2y^2) \, dy$$

$$4\int (y^2-1)^2 \, dy$$

$$4\int y^4-2y^2+1 \, dy$$

$$4\left(\frac{y^5}{5}-\frac{2y^3}{3}+y\right)\Big|_{-1}^{1}$$

$$4\left(\frac{1}{5}-\frac{2}{3}+1-\left(-\frac{1}{5}+\frac{2}{3}-1\right)\right)$$

$$4\left(\frac{2\cdot 3}{5\cdot 3}-\frac{4\cdot 5}{3\cdot 5}+2\right)$$

$$\frac{6}{15}-\frac{20}{15}+\frac{30}{15}$$

$$4\left(\frac{16}{15}\right)=\frac{64}{15}$$