Math 32B Final Counterclockwise

DEREK HU

TOTAL POINTS

82 / 90

QUESTION 1

1 Fubini's Theorem 6 / 6

✓ **+ 6 pts Correct answer (2/3)(e^8 - 1).**

+ **2 pts** (Partial credit) New x limits are 0 to 4.
+ **2 pts** (Partial credit) New y limits are 0 to sqrt(x).
+ **1 pts** (Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 4, sqrt(x) <= y <= 2

QUESTION 2

2 Stokes' Theorem 8 / 8

✓ **+ 8 pts Correct answer 402.**

+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 362.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stoke's theorem.

QUESTION 3

3 Line integral 12 / 12

✓ **+ 4 pts Correct parametrization**

+ **2 pts** Partial credits for parametrization

✓ **+ 4 pts Correct integral formula**

+ **2 pts** Partial credits for integral

✓ **+ 4 pts Correct calculation**

+ **2 pts** Partial credits for calculation
+ **1 pts** Almost makes no sense.
+ **0 pts** Nothing correct

- **1 pts** Tiny calculation error

QUESTION 4

4 Moment of inertia 14 / 14

✓ **+ 1 pts a) Correct limits $$0\leq\rho\leq\frac1{10}$$**

✓ **+ 1 pts a) Correct limits $$0\leq\theta<2\pi$$**

✓ **+ 1 pts a) Correct upper bound $$\phi\leq \pi$$**

✓ **+ 2 pts a) Correct lower bound $$\phi\geq \frac{2\pi}3$$**

✓ **+ 1 pts b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**

✓ **+ 1 pts b) Correct integrand $$5(x^2+y^2)$$ (must substitute $$\delta=5$$ into formula from formula sheet to gain credit)**

✓ **+ 2 pts b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correctly simplifying $$5(x^2+y^2)$$ to $$5\rho^2\sin^2\phi$$**

✓ **+ 2 pts b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correct answer of $$\frac{\pi}{240000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)**

✓ **+ 1 pts Solution thoroughly explained, using full sentences**

✓ **+ 1 pts Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)**

+ **0 pts** No credit due

QUESTION 5

5 Probability **11 / 14**

+ **14 pts** Full points
+ **0 pts** No points
+ **2 pts** Correctly labeled region (all or nothing)

✓ + **3 pts** **Correctly set-up integral (max 6 pts)**

✓ + **2 pts** **Correctly set-up integral**

✓ + **1 pts** **Correctly set-up integral**

✓ + **3 pts** **Evaluation of integral (max 6 pts)**

✓ + **2 pts** **Evaluation of integral**

+ **1 pts** Evaluation of integral

QUESTION 6

6 Divergence Theorem **14 / 14**

✓ + **4 pts** **Correct divergence**

✓ + **7 pts** **Correct parametrization of $$ \mathcal{W}$$**

✓ + **3 pts** **Correct evaluation of correct triple integral (implicit in the grading process was that this rubric item meant that you could have also correctly computed the volume using high school geometry)**

+ **2 pts** Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])

+ **0 pts** No credit

QUESTION 7

7 Vector line integral **9 / 12**

✓ + **4 pts** **Write F as a sum of vortex field and a conservative field**

+ **2 pts** Vortex field has integral 2pi over this C

✓ + **2 pts** **Compute curl_z F_2 or show F_2 is conservative**

✓ + **3 pts** **Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0**

+ **1 pts** Arrive at correct answer, 2pi, by valid method

+ **0 pts** Incorrect

+ **2 pts** Mostly correct argument that integral of F_2 is 0

+ **1 pts** curl_z F_2 minor error

can't apply Green's thm to G

QUESTION 8

8 Surface integral **8 / 10**

✓ + **3 pts** **Decompose flux integral**

+ **1 pts** Partial credit for decomposition

✓ + **2 pts** **Do component integrals**

+ **1 pts** Partial credit for component integrals

+ **1 pts** Combine integrals

✓ + **2 pts** **Used divergence theorem (part (b))**

✓ + **1 pts** **Correct (and justified) div(F) (part (b))**

+ **1 pts** Clear and well-explained solution

+ **0 pts** No credit due

F•N_j=N_1•N_j+...+N_k•N_j≠N_j•N_j, and N_1•N_1+...+N_k•N_k≠F•F. dS does not cancel with the 1/Area; you need to note that the integrand is constant and therefore the integral is equal to F•N_j times Area(S_j).

Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: Derek Hu
SID: 805110848
TA Section: 3C

Time Limit: 180 Minutes

Version 0

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

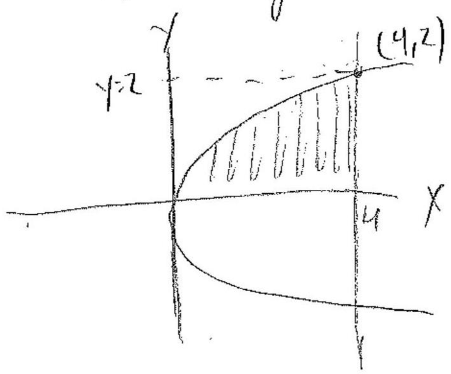
- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x, y, z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x, y, z)\, dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2 + y^2)\, \delta(x, y, z)\, dV$.

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}}\,dx\,dy$.

Sketch region



$0 \le y \le \sqrt{x}$

$0 \le x \le 4$

Switch order of integration

$\int_0^4 \int_0^{\sqrt{x}} e^{x^{3/2}}\,dy\,dx$

$\int_0^4 y e^{x^{3/2}} \Big|_0^{\sqrt{x}}\,dx$

$\int_0^4 \sqrt{x}\, e^{x^{3/2}}\,dx$ let $u = x^{3/2}$ $u(0) = 0$

$du = \frac{3}{2} x^{1/2}\,dx$ $u(4) = 8.$

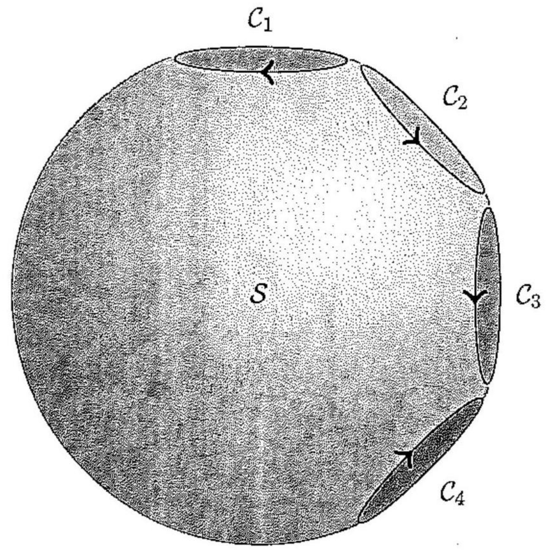
$\frac{2}{3} du = x^{1/2}\,dx$

$\frac{2}{3} \int_0^8 e^u\,du$

$\frac{2}{3} \left(e^u \Big|_0^8 \right)$

$\boxed{\frac{2e^8}{3} - \frac{2}{3}}$

2. (8 points) Let S be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

By Stokes Theorem, $\iint_S \operatorname{curl} \vec{F}\, d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} - \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r}$$

* C_1 & C_4 are oriented positive b/c when we walk along the curve w/ our head pointing outward the surface S is on the left. C_2 & C_3 go the other direction so they are negative

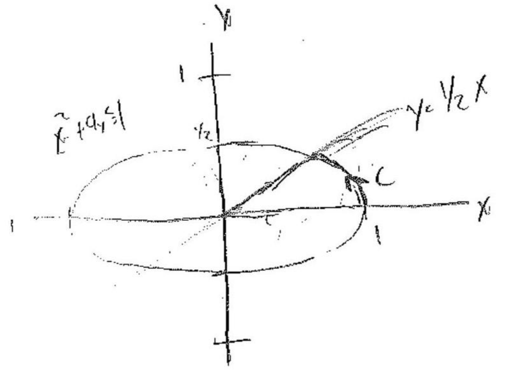
∘ Substitute values

$$20 = \oint_{C_1} \vec{F} \cdot d\vec{r} - 305 - 104 + 27$$

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = 20 + 305 + 104 - 27$$

$$\boxed{\oint_{C_1} \vec{F} \cdot d\vec{r} = 402}$$

3. (12 points) Let $\mathcal{C}$ be the part of the ellipse $x^2 + 4y^2 = 1$ bewteen $y = 0$ and $y = \frac{1}{2}x$ in the first quadrant. Find $\int_{\mathcal{C}} x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$.



Parametrize curve C:

$x = \cos\theta$ $\cos^2\theta + 4\frac{1}{4}\sin^2\theta = 1$ ✓

$y = \frac{1}{2}\sin\theta$

$0 \le \theta \le \pi/4$

@ intersection

$y = \frac{1}{2}x$ $r(\theta) = \langle \cos\theta, \frac{1}{2}\sin\theta \rangle$

$\frac{1}{2}\sin\theta = \frac{1}{2}\cos\theta$ $r'(\theta) = \langle -\sin\theta, \frac{1}{2}\cos\theta \rangle$

$\sin\theta = \cos\theta$

$\theta = \pi/4$ $\|r'(\theta)\| = \sqrt{\sin^2\theta + \frac{1}{4}\cos^2\theta}$

$$\int_C f\, ds = \int_a^b f(r(\theta))\, \|r'(\theta)\|\, d\theta$$

$$= \int_0^{\pi/4} \cos\theta \sqrt{\tfrac{1}{4}\cos^2\theta + \sin^2\theta}\left(\sqrt{\sin^2\theta + \tfrac{1}{4}\cos^2\theta}\right) d\theta$$

$$\int_0^{\pi/4} \cos\theta \left(\tfrac{1}{4}\cos^2\theta + \sin^2\theta\right) d\theta$$

$$\int_0^{\pi/4} \cos\theta \left(\tfrac{1}{4}(1 - \sin^2\theta) + \sin^2\theta\right) d\theta$$

let $u = \sin\theta$ $u(0) = 0$

$du = \cos\theta\, d\theta$ $u(\pi/4) = \frac{\sqrt{2}}{2}$

$$\int_0^{\sqrt{2}/2} \tfrac{1}{4}(1 - u^2) + u^2\, du$$

$$\int_0^{\sqrt{2}/2} \tfrac{3}{4}u^2 + \tfrac{1}{4}\, du$$

$$\tfrac{1}{4}u^3 + \tfrac{1}{4}u \Big|_0^{\sqrt{2}/2}$$

$$\tfrac{1}{4}\left(\tfrac{\sqrt{2}}{2}\right)^3 + \tfrac{1}{4}\left(\tfrac{\sqrt{2}}{2}\right)$$

$$\frac{2\sqrt{2}}{4 \cdot 8} + \frac{\sqrt{2}}{8} = \frac{\sqrt{2}}{16} + \frac{2\sqrt{2}}{16} = \boxed{\frac{3\sqrt{2}}{16}}$$

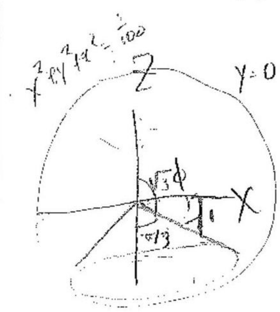
$\rho^2 \le \frac{1}{100}$

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2 + y^2 + z^2 \le \frac{1}{100}$ and $\sqrt{3}z \le -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x, y, z) = 5\,\mathrm{kg\,m^{-3}}$.

   (a) Write $\mathcal{W}$ using spherical coordinates.

   (b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*

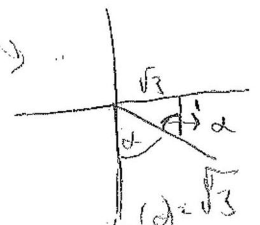
$x^2+y^2+z^2 = \frac{1}{100}$



$y=0$: $\sqrt{3}z \le -x$

$z \le -\frac{1}{\sqrt{3}}x$

a) $\boxed{W: \{0 \le \rho \le \frac{1}{10},\ \frac{2\pi}{3} \le \phi \le \pi,\ 0 \le \theta \le 2\pi}$



$\tan(\alpha) = \sqrt{3}$

$\alpha = \pi/3$ $\quad \phi + \alpha = \pi$

$\phi = 2\pi/3$

b) $I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dx\,dy\,dz$

Convert to spherical coordinates

$x = \rho\cos\theta\sin\phi \qquad dV = \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$

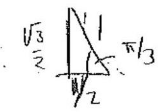
$y = \rho\sin\theta\sin\phi$

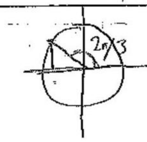
$z = \rho\cos\phi$

$I_z = \iiint_W (\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi)\, 5\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$I_z = \iiint_W (\rho^2\sin^2\phi)\cdot 5\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$\int_0^{2\pi}\int_{2\pi/3}^{\pi}\int_0^{1/10} 5\rho^4\sin^3\phi\, d\rho\, d\phi\, d\theta$





$$\int_0^{2\pi}\int_{2\pi/3}^{\pi} \rho^5 \sin^3\phi \Big|_{\rho=0}^{\rho=1/10} \, d\phi \, d\theta$$

$$\frac{1}{10^5}\int_0^{2\pi}\int_{2\pi/3}^{\pi} \sin^3\phi \, d\phi \, d\theta$$

$$\frac{1}{10^5}\int_0^{2\pi}\int_{2\pi/3}^{\pi} \sin\phi\,(1-\cos^2\phi)\, d\phi \, d\theta$$

let $u = \cos\phi$ $\quad u(2\pi/3) = -\frac{1}{2}$

$du = -\sin\phi \, d\phi$ $\quad u(\pi) = -1$

$$\frac{1}{10^5}\int_0^{2\pi}\int_{-1/2}^{-1} +(-1+u^2)\, du \, d\theta$$

$$\frac{1}{10^5}\int_0^{2\pi} \frac{1}{3}u^3 - u \Big|_{-1/2}^{-1} d\theta$$

$$\frac{2\pi}{10^5}\left(\frac{1}{3}(-1)^3-(-1)-\left(\frac{1}{3}\left(\frac{-1}{2}\right)^3-\left(\frac{-1}{2}\right)\right)\right)$$

$$\frac{2\pi}{10^5}\left(\frac{-1}{3}+1-\left(\frac{-1}{3\cdot 8}+\frac{1}{2}\right)\right)$$

$$\frac{2\pi}{10^5}\left(\frac{2}{3}-\left(\frac{11}{24}\right)\right) = \frac{2\pi}{10^5}\left(\frac{5}{24}\right) = \boxed{\frac{\pi}{24\times 10^4}}\ \text{kg m}^2$$

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \sqrt{3}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{3}{25} \dfrac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.

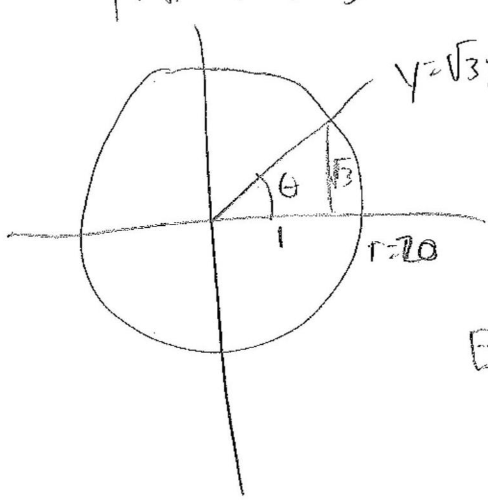
$$E\left[\sqrt{x^2+y^2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2}\; p_{XY} \, dx\, dy$$

$$E\left[\sqrt{x^2+y^2}\right] = \iint_D \sqrt{x^2+y^2}\; p_{XY}\, dA + \iint_{D_1} \sqrt{x^2+y^2}\; p_{XY}\, dA,$$

where D_1 is the region in $\mathbb{R}^2$ that is outside of D. In D_1, $p_{XY} = 0$ so,

$$E\left[\sqrt{X^2+Y^2}\right] = \iint_D \sqrt{X^2+Y^2}\; p_{XY}\, dA$$

First find D



$y = \sqrt{3}x$ $D: \{ 0 \le \theta \le \frac{\pi}{3}$, $0 \le r \le 20 \}$

$X = r\cos\theta$
$Y = r\sin\theta$

$dA = r\, dr\, d\theta$

$$E\left[\sqrt{x^2+y^2}\right] = \int_0^{\pi/3}\int_0^{20} \sqrt{r^2}\left(\frac{3}{25}\frac{r^2\cos^2\theta \cdot r\sin\theta}{(r^2)^{3/2}}\right) r\, dr\, d\theta$$

$$= \int_0^{\pi/3}\int_0^{20} \frac{3}{25} r^2 \cos^2\theta \sin\theta \, dr\, d\theta$$

Fubini $\Rightarrow$

$$\int_0^{20}\int_0^{\pi/3} \frac{3}{25} r^2 \cos^2\theta \sin\theta \, d\theta \, dr$$

$$\int_0^{20} \frac{3}{25} r^2 \int_0^{\pi/3} \cos^2\theta \sin\theta \, d\theta \, dr$$

let $u = \cos\theta$ $u(0) = 1$
$du = -\sin\theta d\theta$ $u(\pi/3) = \frac{1}{2}$

$$\int_0^{20} \frac{3}{25} r^2 \int_1^{1/2} -u^2 \, du \, dr$$

$$\int_0^{20} \frac{3}{25} r^2 \left(-\frac{1}{3} u^3 \Big|_1^{1/2} \right) dr$$

$$\int_0^{20} \frac{3}{25} r^2 \left(-\frac{1}{3 \cdot 8} + \frac{1}{3} \right) dr$$

$$\int_0^{20} \frac{3}{25} \cdot \frac{7}{24} r^2 \, dr$$

$$\frac{7}{25 \cdot 8} \int_0^{20} r^2 \, dr$$

$$= \frac{7}{25 \cdot 8} \left(\frac{1}{3} r^3 \Big|_0^{20} \right)$$

$$\frac{7}{\cancel{200}} \left(\frac{400 \cdot 20}{3} \right) = \boxed{\frac{140}{3}}$$

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $x^2+z^2=1$, $x^2+z^2=9$ and the planes $y=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F}=\left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 3z \right\rangle$ across $\mathcal{S}$. Apply divergence theorem b/c S is a closed surface

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div} F \, dV$$

Convert W to cylindrical coordinates

let $y=y$ $\quad 1\le r\le 3$ $\quad dV = r\,dy\,dr\,d\theta$

$x=r\cos\theta$ $\quad 0\le\theta\le 2\pi$

$z=r\sin\theta$ $\quad r\cos\theta\le y\le 3$

$$\operatorname{div}\vec{F} = \frac{-y(2x)}{(x^2+y^2)^2} + \left(\frac{x(2y)}{(x^2+y^2)^2}\right) + 3 = 3$$

$$= \iiint_W \operatorname{div}\vec{F}\,dV$$

$$= \int_0^{2\pi}\int_1^3\int_{r\cos\theta}^3 3r\,dy\,dr\,d\theta$$

$$\int_0^{2\pi}\int_1^3 9r - 3r^2\cos\theta\,dr\,d\theta$$

$$\int_0^{2\pi} \frac{9}{2}r^2 - \cos\theta\, r^3\Big|_{r=1}^{3}\,d\theta$$

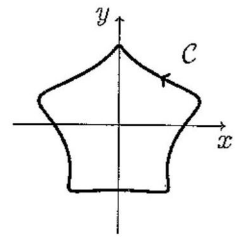
$$\int_0^{2\pi} \frac{9}{2}(9) - 27\cos\theta - \left(\frac{9}{2}-\cos\theta\right)\,d\theta$$

$$\int_0^{2\pi} 36 - 26\cos\theta\,\cancel{d\theta}$$

$$36(2\pi) - \int_0^{2\pi} 26\cos\theta\,d\theta = 0$$

$$= \boxed{72\pi}$$

7. (12 points) Let $\mathcal{C}$ be the curve



Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \cos(x^3) + ye^{xy},\ \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle.$$

(*Hint: Try writing* $\mathbf{F}$ *as a sum of two vector fields that we know how to integrate around* $\mathcal{C}$.)

$$\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle + \left\langle \cos(x^3) + ye^{xy}, e^{e^y} + xe^{xy} \right\rangle$$

$$= \oint_C \vec{F} \cdot d\vec{r} = \oint_C \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot d\vec{r} + \oint_C \left\langle \cos(x^3) + ye^{xy}, e^{e^y} + xe^{xy} \right\rangle \cdot d\vec{r}$$

Let $\vec{G} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

$$\frac{d}{dy}(x^2+y^2)^{-1} = -1(x^2+y^2)^{-2} \cdot 2y$$

$$\operatorname{curl}_z \vec{G} = \frac{d}{dx}\left(\frac{x}{x^2+y^2}\right) - \frac{d}{dy}\left(\frac{-y}{x^2+y^2}\right)$$

$$= \frac{1}{x^2+y^2} + \frac{-2x^2}{(x^2+y^2)^2} + \left(\frac{+1}{x^2+y^2} + \frac{-2y^2}{(x^2+y^2)^2}\right)$$

$$= \frac{2}{x^2+y^2} + \frac{-2x^2-2y^2}{(x^2+y^2)^2}$$

$$= \frac{2}{x^2+y^2} + \frac{-2(x^2+y^2)}{(x^2+y^2)^2}$$

$$= \boxed{0}$$

Now we evaluate

Let $\vec{H} = \langle \cos(x^3) + ye^{xy},\ e^{e^y} + xe^{xy} \rangle$

$$\text{curl}_z \vec{H} = \frac{\partial}{\partial x}\left(e^{e^y} + xe^{xy}\right) - \frac{\partial}{\partial y}\left(\cos(x^3) + ye^{xy}\right)$$

$$= e^{xy} + xye^{xy} - \left(e^{xy} + yxe^{xy}\right) = 0$$

We know that

$\oint_C \vec{F}\, d\vec{r} = \oint_C \vec{G}\, d\vec{r} + \oint_C \vec{H}\, d\vec{r}$, where $\vec{F} = \vec{G} + \vec{H}$, as defined above.

C is a closed curve, so by Green's Theorem...

$\oint_C \vec{G}\, d\vec{r} + \oint_C \vec{H}\, d\vec{r} = \iint_D \text{curl}_z \vec{G}\, dA + \iint_D \text{curl}_z \vec{H}\, dA$

where D is the region enclosed by C.

It was shown that $\text{curl}_z \vec{G} = 0$ and $\text{curl}_z \vec{H} = 0$

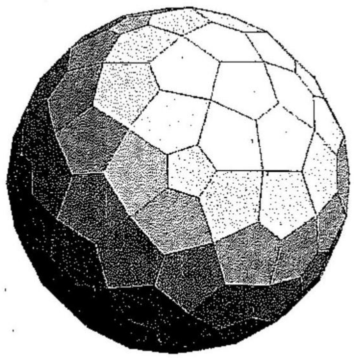
so

$\oint_C \vec{G}\, d\vec{r} + \oint_C \vec{H}\, d\vec{r} = \iint_D 0\, dA + \iint_D 0\, dA = 0$

and finally this leads to the result

$$\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0}$$

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



$dS = \|N\| \, dA$

Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \dots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \dots, k$ define the constant unit vector $\mathbf{a}_j$ so that $\mathbf{a}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\, \mathbf{a}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

a) $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot \vec{a}_1 \, dS_1 + \iint_{S_2} \vec{F} \cdot \vec{a}_2 \, dS_1 + \dots \iint_{S_j} \vec{F} \cdot \vec{a}_j \, dS_j$

where $\vec{a}_j$ is a unit normal vector to surface S_j

$\vec{F} = \vec{N}_1 + \vec{N}_2 + N_3 \dots \vec{N}_k$

$\vec{a}_1 = \frac{\vec{N}_1}{\|N_1\|} + 0\vec{N}_2 + 0\vec{N}_3 \dots 0\vec{N}_k$ (since $\vec{a}_1$ is the unit vector of $\vec{N}$)

so $\vec{P} \cdot \vec{a}_1 = \frac{\vec{N}_1 \cdot \vec{N}_1}{\|N_1\|} = \|N_1\|$

Generalizing ... $\vec{F} \cdot \vec{a}_j = \|N_j\|$

$$\iint_S \vec{F} \cdot d\vec{S} = \sum_{j=1}^{k} \iint_{S_j} \vec{F} \cdot \vec{q_j}\, dS \qquad \vec{q_j} = \frac{\vec{N_j}}{\text{Area}(S_j)}$$

$$= \sum_{j=1}^{k} \iint_{S_j} \vec{F} \cdot \vec{N_j} \left(\frac{1}{\text{Area}(S_j)}\right) dS$$

$$= \sum_{j=1}^{k} \frac{1}{\text{Area}(S_j)} \iint_{S_j} \vec{F} \cdot \vec{N_j}\, dS \qquad \frac{1}{\text{area}} \text{ and } dS \text{ cancel out!}$$

$$= \sum_{j=1}^{k} \vec{F} \cdot \vec{N_j}$$

Since $\vec{F} = \vec{N_1} + \vec{N_2} + \dots \vec{N_j}$

$\vec{F} \cdot \vec{N_1} = \vec{N_1} \cdot \vec{N_1}$ (since the rest of the components of $\vec{N_1}$ are $\vec{0}$)

$$= \sum_{j=1}^{k} \vec{N_j} \cdot \vec{N_j}$$

$$= \vec{N_1} \cdot \vec{N_1} + \vec{N_2} \cdot \vec{N_2} \dots \vec{N_k} + \vec{N_k}$$

$$= \vec{F} \cdot \vec{F} = \boxed{\|F\|^2}$$

b) $\iint_S \vec{F}\, d\vec{S}$: let $f = 1$ $\vec{F} = \vec{F}$ by divergence theorem..

$$\iint_S \vec{F}\, d\vec{S} = \iiint_W \operatorname{div} \vec{F}\, dV$$

$$= \iiint_W \operatorname{div} \vec{N_1}\, dV + \iiint_W \operatorname{div} \vec{N_2}\, dV \dots + \iiint_W \operatorname{div} \vec{N_j}\, dV$$

Since $\vec{N_1}, \vec{N_2}, \dots \vec{N_j}$ are all constant normal vectors $\vec{N_j} = q_j \cdot \text{Area}$

$\operatorname{div}(\vec{N_j}) = 0$ so,

$$\operatorname{div} \vec{F} = \sum_{j=1}^{k} \operatorname{div}(\vec{N_j}) = 0$$

$$\iiint_W \operatorname{div} \vec{F}\, dV = \iiint_W 0\, dV = 0$$

$$\iiint_W \operatorname{div} \vec{F}\, dV = \iint_S \vec{F}\, d\vec{S} = \|\vec{F}\|^2$$

Since $\iiint_W \operatorname{div} \vec{F}\, dV = \iint_S \vec{F}\, d\vec{S}$

$\|\vec{F}\|^2 = 0$

so $\boxed{\vec{F} = \vec{0}}$