# Math 32B Final Clockwise

FRANK XING

TOTAL POINTS

**80 / 90**

QUESTION 1

## 1 Fubini's Theorem 6 / 6

✓ **+ 6 pts Correct answer (2/3)(e^27 - 1).**

+ **2 pts** (Partial credit) New x limits are 0 to 9.
+ **2 pts** (Partial credit) New y limits are 0 to sqrt(x).
+ **1 pts** (Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 9, sqrt(x) <= y <= 3

QUESTION 2

## 2 Stokes' Theorem 8 / 8

✓ **+ 8 pts Correct answer 248.**

+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 208.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stokes theorem.

QUESTION 3

## 3 Line integral 12 / 12

✓ **+ 4 pts Correct parametrization**

+ **2 pts** Partial crerdits for parametrization

✓ **+ 4 pts Correct integral formula**

+ **2 pts** Partial credits for integral

✓ **+ 4 pts Correct calculation and final answer**

+ **2 pts** Partial credits for calculation
+ **1 pts** Almost makes no sense
+ **0 pts** Nothing correct

- **1 pts** Tiny calculation error

QUESTION 4

## 4 Moment of inertia 14 / 14

✓ **+ 1 pts a) Correct limits $$0\leq\rho\leq\frac1{10}$$**

✓ **+ 1 pts a) Correct limits $$0\leq\theta<2\pi$$**

✓ **+ 1 pts a) Correct upper bound $$\phi\leq \pi$$**

✓ **+ 2 pts a) Correct lower bound $$\phi\geq \frac{2\pi}3$$**

✓ **+ 1 pts b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**

✓ **+ 1 pts b) Correct integrand $$3(x^2+y^2)$$ (must substitute $$\delta=3$$ into formula from formula sheet to gain credit)**

✓ **+ 2 pts b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correctly simplifying $$3(x^2+y^2)$$ to $$3\rho^2\sin^2\phi$$**

✓ **+ 2 pts b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correct answer of $$\frac{\pi}{400000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)**

✓ **+ 1 pts Solution thoroughly explained, using full sentences**

+ **1 pts** Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)
+ **0 pts** No credit due

QUESTION 5

## 5 Probability 14 / 14

✓ + **2 pts** Correct limits (max 4 pts)

✓ + **1 pts** Correct limits

✓ + **1 pts** Correct limits

✓ + **2 pts** Correct integrand (max 5 pts)

✓ + **2 pts** correct integrand

✓ + **1 pts** Correct integrand

✓ + **2 pts** Computations (max 5 pts)

✓ + **2 pts** Computations

✓ + **1 pts** Computations

+ **0 pts** No credit due

QUESTION 6

## 6 Divergence Theorem 8 / 14

✓ + **4 pts** Correct divergence

+ **7 pts** Correct parametrization of $$ \mathcal{W}$$

✓ + **3 pts** Correct evaluation of triple integral

+ **2 pts** Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])

+ **0 pts** No credit

+ **1** Point adjustment

QUESTION 7

## 7 Vector line integral 12 / 12

✓ + **4 pts** Write F as a sum of vortex field and a conservative field

✓ + **2 pts** Vortex field has integral 2pi over this C

✓ + **2 pts** Compute curl_z F_2 or show F_2 is conservative

✓ + **3 pts** Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0

✓ + **1 pts** Arrive at correct answer, 2pi, by valid method

+ **0 pts** Incorrect

+ **2 pts** Mostly correct argument that integral of F_2 is 0

+ **1 pts** curl_z F_2 minor error

QUESTION 8

## 8 Surface integral 6 / 10

✓ + **3 pts** Decompose flux integral

+ **1 pts** Partial credit for decomposition

✓ + **2 pts** Do component integrals

+ **1 pts** Partial credit for component integrals

✓ + **1 pts** Combine integrals

+ **2 pts** Used divergence theorem (part (b))

+ **1 pts** Correct (and justified) div(F) (part (b))

+ **1 pts** Clear and well-explained solution

+ **0 pts** No credit due

F is not Area(S) times anything sensible.

W is not necessarily symmetric, and is not a sphere.

**Math 32B - Lectures 3 & 4**
**Winter 2019**
**Final Exam**
**3/17/2019**

**Name:** Frank Xing
**SID:** 905-164-685
**TA Section:** Discussion 4B

**Time Limit:** 180 Minutes

**Version** 0

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This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

## Mechanics formulas

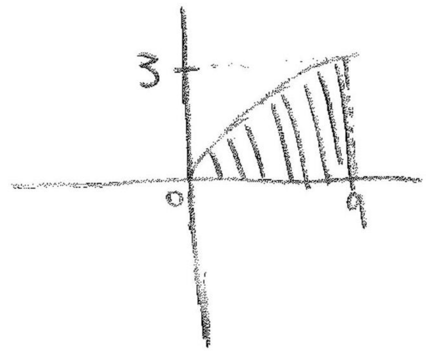
- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$.
  - The $y$-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$.
  - The $x$-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the $x$-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$.
  - The moment of inertia about the $y$-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x,y,z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x,y,z)\,dV$.
  - The $yz$-moment is $M_{yz} = \iiint_{\mathcal{W}} x\,\delta(x,y,z)\,dV$.
  - The $xz$-moment is $M_{zx} = \iiint_{\mathcal{W}} y\,\delta(x,y,z)\,dV$.
  - The $xy$-moment is $M_{xy} = \iiint_{\mathcal{W}} z\,\delta(x,y,z)\,dV$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}, z_{\text{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the $x$-axis is $I_x = \iiint_{\mathcal{W}} (y^2+z^2)\,\delta(x,y,z)\,dV$.
  - The moment of inertia about the $y$-axis is $I_y = \iiint_{\mathcal{W}} (x^2+z^2)\,\delta(x,y,z)\,dV$.
  - The moment of inertia about the $z$-axis is $I_z = \iiint_{\mathcal{W}} (x^2+y^2)\,\delta(x,y,z)\,dV$.

## Probability formulas

- If a continuous random variable $X$ has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables $X, Y$ have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^3 \int_{y^2}^9 e^{x^{\frac{3}{2}}}\, dx dy$.

We draw out the region D:




Then, we can change order with Fubini's theorem.

$$\int_0^3 \int_{y^2}^9 e^{x^{3/2}}\, dx\, dy = \int_0^9 \int_0^{\sqrt{x}} e^{x^{3/2}}\, dy\, dx$$

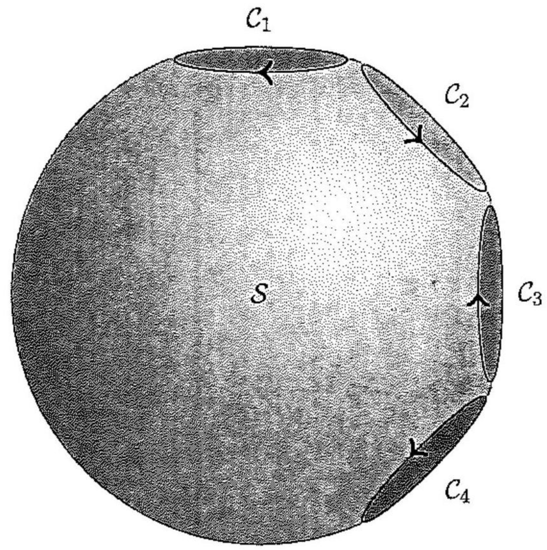
$$= \int_0^9 \sqrt{x}\, e^{x^{3/2}}\, dx$$

$$= \left(\frac{2}{3} e^{x^{3/2}}\right)\Big|_0^9$$

$$= \frac{2}{3} e^{27} - \frac{2}{3}$$

$$= \frac{2}{3}\left(e^{27} - 1\right)$$

2. (8 points) Let $\mathcal{S}$ be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:




Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

We know the total flux on the sphere should be 0 since there are normals pointing in all direction.

By Stoke's theorem:

$$0 = \iint_S \operatorname{curl} F \cdot ds - \oint_{C_1} F \cdot dr + \oint_{C_2} F \cdot dr - \oint_{C_3} F \cdot dr + \oint_{C_4} F \cdot dr$$

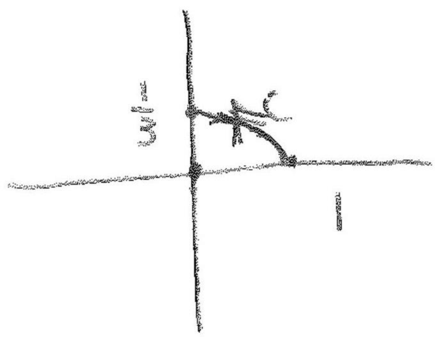
$$0 = 20 - \oint_{C_1} F \cdot dr + 305 - 104 + 27$$

$$0 = 248 - \oint_{C_1} F \cdot dr$$

$$\oint_{C_1} F \cdot dr = 248$$

3. (12 points) Let $\mathcal{C}$ be the part of the ellipse $x^2+9y^2=1$ bewteen $y=0$ and $y=\frac{1}{3}x$ in the first quadrant. Find $\int_{\mathcal{C}} x\sqrt{\frac{1}{9}x^2+9y^2}\,ds$.

We sketch the curve:




We plug in $y=\frac{1}{3}x$ into the equation:

$$(3y)^2+9y^2=1$$
$$18y^2=1$$
$$y=\frac{1}{\sqrt{18}}$$

Then, we can parameterize the curve

$$\langle \sqrt{1-9y^2},\ y \rangle$$

from $0 \le y \le \frac{1}{\sqrt{18}}$.

Then, we find $r'(y) = \left\langle \frac{-9y}{\sqrt{(1-9y^2)}},\ 1 \right\rangle$ and $\|r'(y)\| = \sqrt{\frac{72y^2+1}{1-9y^2}}$

We compute $\int_{\mathcal{C}} f \cdot d\vec{s} = \int_{\mathcal{C}} f(r(y))\|r'(y)\|\,dy$

$$= \int_0^{\frac{1}{\sqrt{18}}} \sqrt{1-9y^2}\sqrt{\frac{72y^2+1}{1-9y^2}}\sqrt{\frac{1}{9}(1-9y^2)+9y^2}\ dy$$

$$= \int_0^{\frac{1}{\sqrt{18}}} \sqrt{72y^2+1}\ \sqrt{\frac{1}{9}+8y^2}\ dy$$

$$= \int_0^{\frac{1}{\sqrt{18}}} 3\sqrt{8y^2+\frac{1}{9}}\ \sqrt{8y^2+\frac{1}{9}}\ dy$$

$$= \int_0^{\frac{1}{\sqrt{18}}} 24y^2+\frac{1}{3}\,dy = \left(8y^3+\frac{1}{3}y\right)\Big|_0^{\frac{1}{\sqrt{18}}}$$

$$= \frac{8}{\sqrt{18^3}} + \frac{1}{3\sqrt{18}}$$

$$= \frac{8}{18\sqrt{18}} + \frac{1}{3\sqrt{18}}$$

$$= \frac{8}{54\sqrt{2}} + \frac{1}{9\sqrt{2}}$$

$$= \frac{14}{54\sqrt{2}} = \frac{7}{27\sqrt{2}}$$

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2+y^2+z^2 \leq \frac{1}{100}$ and $\sqrt{3}z \leq -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x,y,z) = 3\,\mathrm{kg\,m^{-3}}$.
   (a) Write $\mathcal{W}$ using spherical coordinates.
   (b) Find the moment of inertia of $\mathcal{W}$ about the $z$-axis. *(Do not forget to use the correct units.)*

(a) With sphereical coordinates,

we see as $<\rho\sin\phi\cos\theta, \rho\sin\theta\sin\phi, \rho\cos\phi>$.

We know $x^2+y^2+z^2 \leq \frac{1}{100}$

so $\rho^2 \leq \frac{1}{100}$

and $0 \leq \rho \leq \frac{1}{10}$

Additionally,

$\sqrt{3}\rho\cos\phi \leq -\rho\sin\phi$ — $\sin\phi$ is positive between 0 and $\pi$.

$\sqrt{3}\cos\phi \leq -\sin\phi \rightarrow \frac{\cos\phi}{\sin\phi} \leq \frac{-\sqrt{3}}{3}$

The only place possible angle for this to happen is

$\frac{2\pi}{3} \leq \phi \leq \pi$.

Therefore, the bounds are $0 \leq \theta \leq 2\pi$,

$\frac{2\pi}{3} \leq \phi \leq \pi$ and $0 \leq \rho \leq \frac{1}{10}$

(b) We compute $\int_0^{2\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^{\frac{1}{10}} 3\rho^2\sin^2\phi\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

Inertia = $\iiint_W (x^2+y^2)\,\sigma(x,y,z)\,dV$

$= \int_0^{2\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^{\frac{1}{10}} 3\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi}\int_{\frac{2\pi}{3}}^{\pi} \left(\frac{3}{5}\rho^5\sin^3\phi\right)\Big|_0^{\frac{1}{10}}\,d\phi\,d\theta$

$$= \int_0^{2\pi} \int_{\frac{2\pi}{3}}^{\pi} \left( \frac{3}{5 \cdot 10^5} \sin^3\phi \right) d\phi \, d\theta$$

$$= \frac{3}{5 \cdot 10^5} \int_0^{2\pi} \int_{\frac{2\pi}{3}}^{\pi} (1 - \cos^2\phi) \sin\phi \, d\phi \, d\theta$$

$$= \frac{3}{5 \cdot 10^5} \int_0^{2\pi} \left( \frac{\cos^3\phi}{3} - \cos\phi \right) \Big|_{\frac{2\pi}{3}}^{\pi} d\theta$$

$$= \frac{3}{5 \cdot 10^5} \int_0^{2\pi} \frac{5}{24} \, d\theta$$

$$= \frac{3}{5 \cdot 10^5} \cdot \frac{10\pi}{24} = \frac{2\pi}{8 \cdot 10^5} = \frac{2\pi}{800000}$$

$$= \frac{\pi}{400000} \text{ kg} \cdot \text{m}^2$$

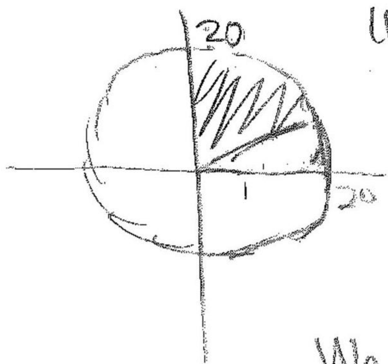
5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \frac{1}{\sqrt{3}}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables $(X, Y)$ with joint probability density

$$p_{X,Y}(x,y) = \begin{cases} \frac{3}{175}\frac{xy^2}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.

We sketch the region

(Diagram not to scale)




The $E[\sqrt{X^2+Y^2}]$

$$= \iint_D \sqrt{x^2+y^2}\,\frac{3}{175}\frac{xy^2}{(x^2+y^2)^{3/2}}\,dy\,dx$$

We can swap to polar coordinates

$$= \iint_D r\,\frac{3}{175}\,\frac{r^3\cos\theta\sin^2\theta}{r^3}\,r\,dr\,d\theta$$

From the region, we see

$0 \le r^2 \le 400$

$0 \le r \le 20$

We also know from the diagram

Therefore $\frac{\pi}{6} \le \theta \le \frac{\pi}{2}$

We can compute $\displaystyle\int_{\pi/6}^{\pi/2}\int_0^{20} \frac{3}{175} r^2\cos\theta\sin^2\theta\,dr\,d\theta$

$$= \frac{3}{175}\int_{\pi/6}^{\pi/2} \frac{20^3}{3}\cos\theta\sin^2\theta\,d\theta$$

$$= \frac{3\cdot 20^3}{175\cdot 3}\left(\frac{\sin^3\theta}{3}\right)\Big|_{\pi/6}^{\pi/2}$$

$$= \frac{3 \cdot 20^3}{175 \cdot 3}\left(\frac{1}{3} - \frac{1}{24}\right)$$

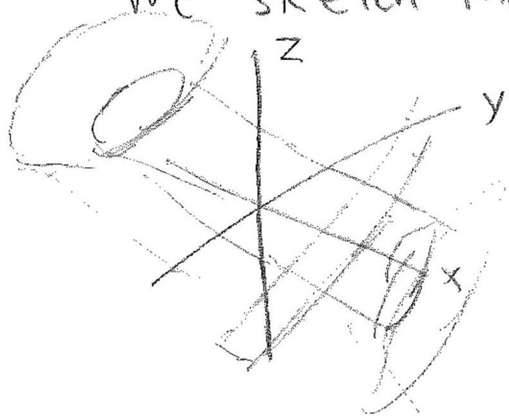
$$= \frac{3 \cdot 20^3 \cdot 7}{175 \cdot 24 \cdot 3} - \frac{7 \cdot 20^3}{175 \cdot 24}$$

$$= \frac{20^3 \cdot 7}{4200}$$

$$= \frac{56000}{4200} = \frac{280}{21} = \frac{40}{3}$$

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $y^2+z^2=1$, $y^2+z^2=9$ and the planes $x=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F}=\left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 2z \right\rangle$ across $\mathcal{S}$.

We sketch the region




Using the divergence theorem,

we know

$$\iiint_W \operatorname{div}\vec{F} = \iint_S \vec{F}\cdot d\vec{S}.$$

We compute $\operatorname{div}\vec{F}$

$$= \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 2$$

$$= 2.$$

Then, we find the bounds for $W$

by changing to cylindrical coordinate

$\langle x, r\sin\theta, r\cos\theta\rangle$.

We know $r\sin\theta \le x \le 3$ and $0 \le r \le 1$

and $0 \le \theta \le 2\pi$.

Therefore, $\int_0^{2\pi}\int_0^1\int_{r\sin\theta}^3 2r \, dx\, dr\, d\theta$

$$= \int_0^{2\pi}\int_0^1 6r - 2r^2\sin\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} \left(3r^2 - \frac{2}{3}r^3\sin\theta\right)\Big|_0^1 \, d\theta$$

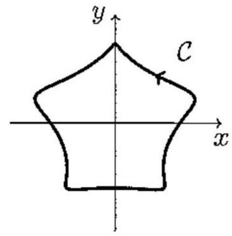
$$= \int_0^{2\pi} 3 - \frac{2}{3}\sin\theta \, d\theta$$

$$= 6\pi\left(3\theta + \frac{2}{3}\cos\theta\right)\Big|_0^{2\pi}$$

$$= 6\pi + \frac{2}{3} - \frac{2}{3}$$

$$= 6\pi.$$

7. (12 points) Let $\mathcal{C}$ be the curve




Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \sin(x^5) + 2ye^{2xy},\ \frac{x}{x^2+y^2} + e^{\cos(y)} + 2xe^{2xy} \right\rangle.$$

(*Hint: Try writing* **F** *as a sum of two vector fields that we know how to integrate around* $\mathcal{C}$.)

We can write $\vec{F}$

as $\left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

$+ \langle \sin(x)^5 + 2ye^{2xy}, e^{\cos y} + 2xe^{2xy} \rangle$

We alreay know $\oint_c \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle \cdot dr$

$= 2\pi$ since it loops the origin once (property of vortex field)

Then, we find the curl for the second function

$\text{curl } \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

$= 2e^{2xy} + 4xye^{2xy} - 2e^{2xy} - 4xye^{2xy}$

$= 0.$

Since $\vec{F}$ is also simply connected, $\vec{F}$ is conservative and $\oint_c \vec{F} \cdot d\vec{r}$ over a closed loop is 0.

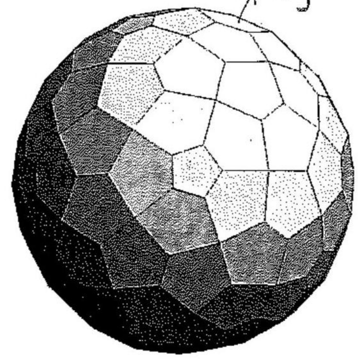
Adding the two values up,

we get $\oint_C F \cdot d\vec{r} = 2\pi + 0$

$= 2\pi$.

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example

$b_j$



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of $k$ planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{b}_j$ so that $\mathbf{b}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{b}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

(a) Since $N_j = \text{Area}(S_j)\, b_j$

We know $N_j =$ The surface area $\cdot$ normal vector

$= \iint_{S_j} dS$

We also know, $F =$ The sum of $\iint_{S_j} dS$ for all $j$'s (from given)

This means $F =$ The total surface area $\cdot$ by all corresponding normal vectors

$= \iint_S dS$

Since $\|F\|^2 = F \cdot F$

$= F \cdot \iint_S dS$

$= \iint_S F \cdot dS$ (Since $F$ is a constant)

Therefore, the statement is true

(b) If $F=0$, then $\iint_S F \cdot dS = 0$

We look at the case $\iint_S dS$ and

observed it is made up of

$\iint_{S_1} dS + \iint_{S_2} dS \cdots \iint_{S_k} dS$.

Looking at the figure, we note that each surface has a corresponding area on the other side with an unit normal in the opposite direction (since the whole object is a sphere with normal vectors in all directions). Therefore, the sum would be a net total of 0.

This means $\iint_S dS$ would be 0,

leading to $\iint_S F \cdot dS = 0$, $\|F\|^2 = 0$

and $F=0$.