Math 32B Final Counterclockwise

NIKIL SELVAM

TOTAL POINTS

90 / 90

QUESTION 1

1 Fubini's Theorem 6 / 6

✓ **+ 6 pts Correct answer (2/3)(e^8 - 1).**

+ **2 pts** (Partial credit) New x limits are 0 to 4.
+ **2 pts** (Partial credit) New y limits are 0 to sqrt(x).
+ **1 pts** (Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 4, sqrt(x) <= y <= 2

QUESTION 2

2 Stokes' Theorem 8 / 8

✓ **+ 8 pts Correct answer 402.**

+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 362.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stoke's theorem.

QUESTION 3

3 Line integral 12 / 12

✓ **+ 4 pts Correct parametrization**

+ **2 pts** Partial credits for parametrization

✓ **+ 4 pts Correct integral formula**

+ **2 pts** Partial credits for integral

✓ **+ 4 pts Correct calculation**

+ **2 pts** Partial credits for calculation
+ **1 pts** Almost makes no sense.
+ **0 pts** Nothing correct

- **1 pts** Tiny calculation error

QUESTION 4

4 Moment of inertia 14 / 14

✓ **+ 1 pts a) Correct limits $$0\leq\rho\leq\frac1{10}$$**

✓ **+ 1 pts a) Correct limits $$0\leq\theta<2\pi$$**

✓ **+ 1 pts a) Correct upper bound $$\phi\leq \pi$$**

✓ **+ 2 pts a) Correct lower bound $$\phi\geq \frac{2\pi}3$$**

✓ **+ 1 pts b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**

✓ **+ 1 pts b) Correct integrand $$5(x^2+y^2)$$ (must substitute $$\delta=5$$ into formula from formula sheet to gain credit)**

✓ **+ 2 pts b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correctly simplifying $$5(x^2+y^2)$$ to $$5\rho^2\sin^2\phi$$**

✓ **+ 2 pts b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correct answer of $$\frac{\pi}{240000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)**

✓ **+ 1 pts Solution thoroughly explained, using full sentences**

✓ **+ 1 pts Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)**

+ **0 pts** No credit due

QUESTION 5

5 Probability 14 / 14

✓ + **14 pts** **Full points**

+ **0 pts** No points
+ **2 pts** Correctly labeled region (all or nothing)
+ **3 pts** Correctly set-up integral (max 6 pts)
+ **2 pts** Correctly set-up integral
+ **1 pts** Correctly set-up integral
+ **3 pts** Evaluation of integral (max 6 pts)
+ **2 pts** Evaluation of integral
+ **1 pts** Evaluation of integral

QUESTION 6

6 Divergence Theorem 14 / 14

✓ + **4 pts** **Correct divergence**

✓ + **7 pts** **Correct parametrization of $$ \mathcal{W}$$**

✓ + **3 pts** **Correct evaluation of correct triple integral (implicit in the grading process was that this rubric item meant that you could have also correctly computed the volume using high school geometry)**

✓ + **2 pts** **Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])**

+ **0 pts** No credit

QUESTION 7

7 Vector line integral 12 / 12

✓ + **4 pts** **Write F as a sum of vortex field and a conservative field**

✓ + **2 pts** **Vortex field has integral 2pi over this C**

✓ + **2 pts** **Compute curl_z F_2 or show F_2 is conservative**

✓ + **3 pts** **Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0**

✓ + **1 pts** **Arrive at correct answer, 2pi, by valid method**

+ **0 pts** Incorrect
+ **2 pts** Mostly correct argument that integral of F_2 is 0
+ **1 pts** curl_z F_2 minor error

QUESTION 8

8 Surface integral 10 / 10

✓ + **3 pts** **Decompose flux integral**

+ **1 pts** Partial credit for decomposition

✓ + **2 pts** **Do component integrals**

+ **1 pts** Partial credit for component integrals

✓ + **1 pts** **Combine integrals**

✓ + **2 pts** **Used divergence theorem (part (b))**

✓ + **1 pts** **Correct (and justified) div(F) (part (b))**

✓ + **1 pts** **Clear and well-explained solution**

+ **0 pts** No credit due

You don't need to expand the F; everything works fine without it.

Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: NIKIL ROASHAN SELVAM
[redacted]
TA Section: 4F

Time Limit: 180 Minutes

Version 0

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

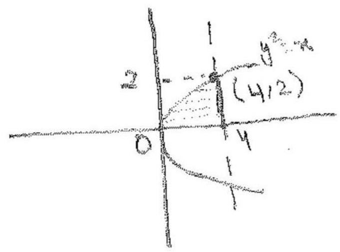
- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x, y, z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x, y, z)\, dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2 + y^2)\, \delta(x, y, z)\, dV$.

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}}\, dxdy$.

Given domain $D = \{ 0 \le y \le 2 \ , \ y^2 \le x \le 4 \}$



It can be re-written as $D : \{ 0 \le x \le 4 , \ 0 \le y \le \sqrt{x} \}$

$\therefore$ By Fubini's thm,

$$\int_0^2 \int_{y^2}^4 e^{x^{3/2}}\, dxdy = \int_0^4 \int_0^{\sqrt{x}} e^{x^{3/2}}\, dydx$$

$$= \int_0^4 e^{x^{3/2}} [y]_0^{\sqrt{x}}\, dx$$

$$= \int_0^4 \sqrt{x}\, e^{x^{3/2}}\, dx$$

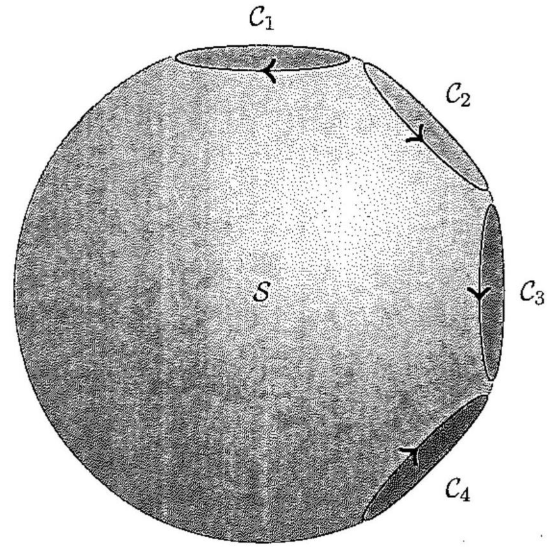
$$\left[\text{let } \quad x^{3/2} = t \quad \begin{array}{l} 0 \le x \le 4 \\ \Rightarrow 0 \le t \le 8 \end{array} \atop \quad \tfrac{3}{2}\sqrt{x}\,dx = dt \right]$$

$$= \frac{2}{3} \int_0^8 e^t\, dt$$

$$= \frac{2}{3} \left[e^t \right]_0^8$$

$$= \frac{2}{3} \left[e^8 - 1 \right]$$

2. (8 points) Let $\mathcal{S}$ be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

Given, S has outward pointing normal.

By Stokes thm,

$\iint_S \text{curl}\vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$, where C is the boundary curve of S.

For given surface,

we have 4 parts of the boundary curve

C_1, C_2, C_3, C_4

Applying Stokes' thm with correct signs for orientation [∵ S has outward normal],

we get

$$\iint_S \operatorname{curl} \vec{F} \cdot \vec{dS} = \oint_{C_1} \vec{F} \cdot \vec{dr} - \oint_{C_2} \vec{F} \cdot \vec{dr} - \oint_{C_3} \vec{F} \cdot \vec{dr} + \oint_{C_4} \vec{F} \cdot \vec{dr}$$

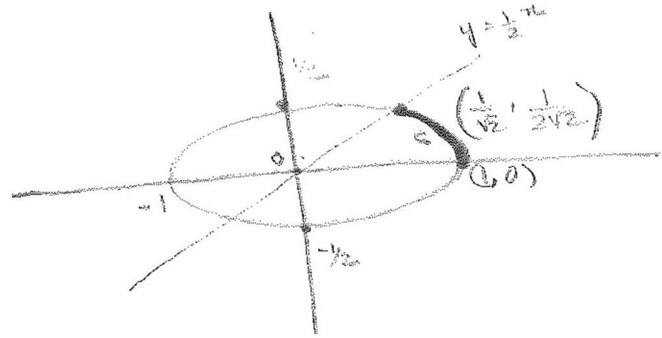
$$20 = \oint_{C_1} \vec{F} \cdot \vec{dr} - 305 - 104 + 27$$

$$-7 = \oint_{C_1} \vec{F} \cdot \vec{dr} - 409$$

$$\oint_{C_1} \vec{F} \cdot \vec{dr} = 409 - 7 = 402$$

$$\therefore \oint_{C_1} \vec{F} \cdot \vec{dr} = 402$$

3. (12 points) Let $\mathcal{C}$ be the part of the ellipse $x^2 + 4y^2 = 1$ bewteen $y = 0$ and $y = \frac{1}{2}x$ in the first quadrant. Find $\int_{\mathcal{C}} x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$.



Point of intersection of line and ellipse:

$y = \frac{1}{2}x$

$\Rightarrow x = 2y$

$x^2 + 4y^2 = 1$

$4y^2 + 4y^2 = 1$

$y^2 = \frac{1}{8} \Rightarrow y = \frac{1}{2\sqrt{2}}$

$x = \frac{1}{\sqrt{2}}$

$\therefore$ C is part of given ellipse from $(1, 0)$ to $\left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)$

Parametrize C by $\vec{r}(t) = \left\langle \cos t, \frac{1}{2}\sin t \right\rangle$

$0 \le t \le \frac{\pi}{4}$ $\left[\because (1,0) \text{ to } \left(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}\right)\right]$

$\therefore \vec{r}'(t) = \left\langle -\sin t, \frac{1}{2}\cos t \right\rangle$

$\|\vec{r}'(t)\| = \sqrt{\sin^2 t + \frac{1}{4}\cos^2 t}$

$\therefore \int_C x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$

$= \int_0^{\pi/4} \cos t \sqrt{\frac{1}{4}\cos^2 t + \sin^2 t}\ \|\vec{r}'(t)\|\, dt$

$= \int_0^{\pi/4} \cos t \left(\frac{1}{4}\cos^2 t + \sin^2 t\right) dt$

$$= \int_0^{\pi/4} \cos t \left[\frac{1}{4}\cos^2 t + \frac{1}{4}\sin^2 t + \frac{3}{4}\sin^2 t \right] dt$$

$$= \int_0^{\pi/4} \cos t \left[\frac{1}{4} + \frac{3}{4}\sin^2 t \right] dt$$

$$= \frac{1}{4}\int_0^{\pi/4} \cos t \, dt + \frac{3}{4}\int_0^{\pi/4} \cos t \sin^2 t \, dt$$

$$= \frac{1}{4}\left[\sin t \right]_0^{\pi/4} + \frac{3}{4}\int_0^{\pi/4} \cos t \sin^2 t \, dt$$

$$= \frac{1}{4\sqrt{2}} + \frac{3}{4}\int_0^{\pi/4} \cos t \sin^2 t \, dt$$

$$\left[\begin{array}{ll} \text{let } \sin t = p & \Rightarrow 0 \le t \le \frac{\pi}{4} \\ \cos t \, dt = dp & 0 \le p \le \frac{1}{\sqrt{2}} \end{array} \right]$$

$$= \frac{1}{4\sqrt{2}} + \frac{3}{4}\int_0^{1/\sqrt{2}} p^2 \, dp$$

$$= \frac{1}{4\sqrt{2}} + \frac{3}{4}\left[\frac{p^3}{3} \right]_0^{1/\sqrt{2}}$$

$$= \frac{1}{4\sqrt{2}} + \frac{1}{4} \cdot \frac{1}{2\sqrt{2}}$$

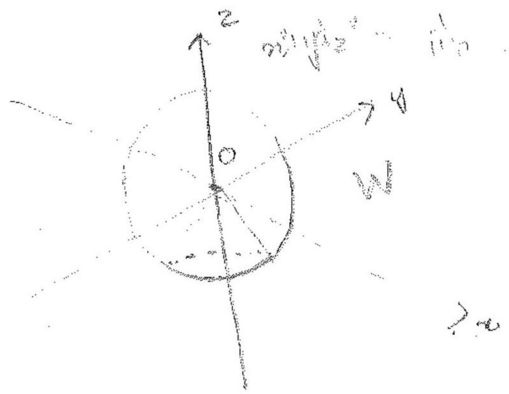
$$= \frac{1}{4\sqrt{2}}\left[1 + \frac{1}{2} \right]$$

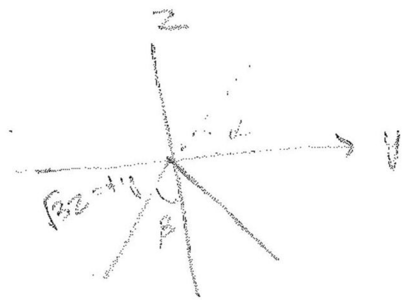
$$= \frac{3}{8\sqrt{2}}$$

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2 + y^2 + z^2 \le \frac{1}{100}$ and $\sqrt{3}z \le -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x, y, z) = 5\,\mathrm{kg\,m^{-3}}$.

   (a) Write $\mathcal{W}$ using spherical coordinates.

   (b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*





slope of line $= \frac{1}{\sqrt{3}}$

$\Rightarrow \tan\theta = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \frac{\pi}{6}$

$\Rightarrow \beta = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$

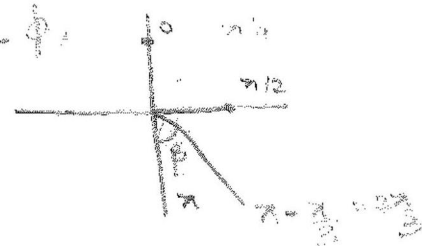
a) Parametrize W using spherical coordinates.

$\langle \rho\cos\theta\sin\phi,\ \rho\sin\theta\sin\phi,\ \rho\cos\phi \rangle$

$0 \le \rho \le \frac{1}{10}$ $[x^2+y^2+z^2 \le \frac{1}{100} \Rightarrow \rho \le \frac{1}{10}]$

$0 \le \theta \le 2\pi$ [Symmetry about z axis]

$\frac{2\pi}{3} \le \phi \le \pi$ [∵ For ϕ,



]

b) Given density of solid $\delta(x,y,z) = 5\ \mathrm{kg\,m^{-3}}$.

Moment of inertia about z-axis is

$$I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dV$$

$$= \int_0^{2\pi}\int_{2\pi/3}^{\pi}\int_0^{1/10} \rho^2\sin^2\phi \cdot 5 \cdot \rho^2\sin\phi \; d\rho\, d\phi\, d\theta \quad \left[\begin{matrix}\text{where jacobian is} \\ \rho^2\sin\phi\end{matrix}\right]$$

$$= 5 \cdot 2\pi \int_{2\pi/3}^{\pi}\int_0^{1/10} \rho^4 \sin^3\phi \, d\rho\, d\phi$$ [$\because \theta$ is completely independent of all other variables in the integral]

$$= 10\pi \int_{2\pi/3}^{\pi} \sin^3\phi \left[\frac{\rho^5}{5}\right]_0^{1/10} d\phi$$

$$= \frac{10\pi}{5}\cdot\left(\frac{1}{10}\right)^5 \int_{2\pi/3}^{\pi} \sin^3\phi\, d\phi$$

$$\left[\begin{aligned} \int \sin^3 t\,dt = \int \sin t(1-\cos^2 t)\,dt &= \int \sin t\,dt - \int \cos^2 t \sin t\,dt \\ &= -\cos t + \frac{\cos^3 t}{3} + C \end{aligned}\right]$$

$$= 2\pi\cdot\left(\frac{1}{10}\right)^5 \left[\frac{\cos^3\phi}{3} - \cos\phi\right]_{2\pi/3}^{\pi}$$

$$= 2\pi\cdot\left(\frac{1}{10}\right)^5 \left[\frac{-1}{3} - (-1) - \left(\frac{-1}{24} - \left(-\frac{1}{2}\right)\right)\right]$$

$$= \frac{2\pi}{(10)^5}\left[-\frac{1}{3} + 1 - \left(\frac{-1}{24} + \frac{1}{2}\right)\right]$$

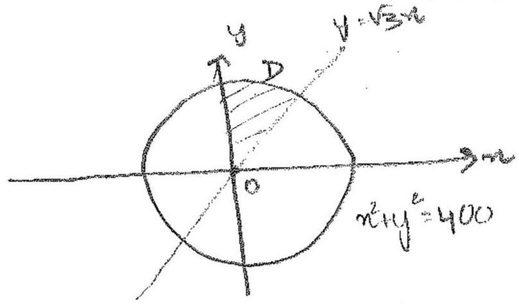
$$= \frac{2\pi}{(10)^5}\left[\frac{2}{3} - \frac{11}{24}\right]$$

$$= \frac{2\pi}{(10)^5}\cdot\frac{5}{24} = \boxed{\frac{\pi}{24(10^4)}\ \text{Kg m}^2}$$

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \sqrt{3}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

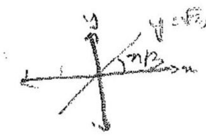
$$p_{X,Y}(x,y) = \begin{cases} \frac{3}{25} \frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.



D can be expressed in polar co-ordinates as $(x,y) = (r\cos\theta, r\sin\theta)$.

$D = \{0 \le r \le 20,\ \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$

[

$m = \sqrt{3}$ $\Rightarrow \theta = \pi/3$]

$$\mathbb{E}\left[\sqrt{X^2+Y^2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2}\ p(x,y)\,dx\,dy$$

$$= \iint_D \sqrt{x^2+y^2}\ \frac{3}{25}\frac{x^2y}{(x^2+y^2)^{3/2}}\,dx\,dy$$

$[\because p_{x,y}(x,y) = 0 \ \forall\ (x,y) \notin D]$

$$= \iint_D \frac{3}{25}\frac{x^2y}{x^2+y^2}\,dx\,dy$$

[Changing to polar co-ordinates]

$$= \frac{3}{25}\int_{\pi/3}^{\pi/2}\int_0^{20} \frac{r^2\cos^2\theta\, r\sin\theta}{r^2} \cdot r\,dr\,d\theta$$

[where Jacobian is r]

$$= \frac{3}{25} \int_{\pi/3}^{\pi/2} \int_0^{20} r^2 \cos^2\theta \sin\theta \, dr \, d\theta$$

$$= \frac{\cancel{3}}{25} \int_{\pi/3}^{\pi/2} \cos^2\theta \sin\theta \left[\frac{r^3}{\cancel{3}} \right]_0^{20} d\theta$$

$$= \frac{(20)^3}{25} \int_{\pi/3}^{\pi/2} \cos^2\theta \sin\theta \, d\theta$$

$$\left[\begin{array}{l} \int \cos^2 t \sin t \, dt \quad , \text{ let } \cos t = p \quad -\sin t \, dt = dp \\ = -\int p^2 \, dp \\ = -\frac{p^3}{3} + c \; = \; -\frac{\cos^3 t}{3} + c \end{array} \right]$$

$$= -\frac{400 \cdot 20}{25} \left[\frac{\cos^3\theta}{3} \right]_{\pi/3}^{\pi/2}$$

$$= -16 \cdot 20 \left[-\frac{\cos^3 \frac{\pi}{3}}{3} \right]$$

$$= \frac{320}{3} \cdot \left(\frac{1}{2}\right)^3$$

$$= \frac{\cancel{320}^{40}}{3} \cdot \frac{1}{\cancel{8}}$$

$$= \boxed{\frac{40}{3}}$$

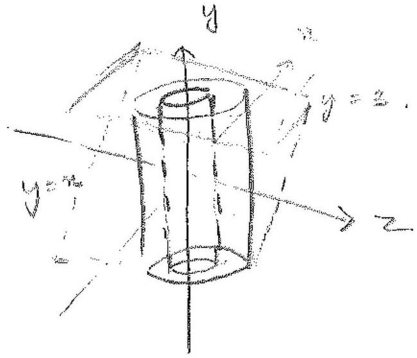
$\therefore$ Expected distance of shot is $\frac{40}{3}$ units [Presumably metres]

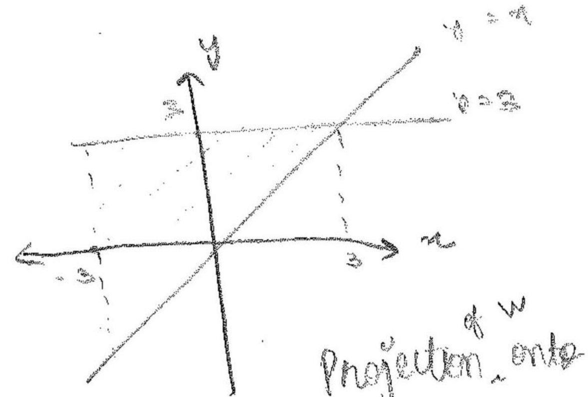
6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $x^2+z^2=1$, $x^2+z^2=9$ and the planes $y=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F} = \left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 3z \right\rangle$ across $\mathcal{S}$.

Given $\vec{F} = \left\langle \frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}, 3z \right\rangle$

Now, $\operatorname{div} \vec{F} = \frac{-y}{(x^2+y^2)^2} \cdot 2x - \left(\frac{-x \cdot 2y}{x^2+y^2} \right) + 3$

$= 3$





Region W can be parametrized in cylindrical co-ordinates as

$G(r, \theta, h) = \langle r\cos\theta, h, r\sin\theta \rangle = \langle x, y, z \rangle$

$1 \le r \le 3$ $[\because 1 \le x^2+z^2 \le 9]$

$0 \le \theta \le 2\pi$ (Symmetry about y-axis)

$r\cos\theta \le h \le 3$

$\left[\begin{array}{l} \because x \le y \le 3 \\ x^2+z^2 \le 9 \\ \Rightarrow x^2 \le 9 \\ \Rightarrow x \le 3 \\ \therefore \text{It is bounded above by } y=3 \text{ and below by } y=x \end{array} \right]$

S is the boundary surface with outward normal of region W.

$\therefore$ By divergence thm,

Flux of vector field $\vec{F}$ across S,

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}\vec{F}\, dV$$

$$= \iiint_W 3\, dV$$

$$= 3\int_0^{2\pi}\int_1^3\int_{r\cos\theta}^{3} r\, dh\, dr\, d\theta \quad \left[\begin{array}{c}\text{where Jacobian} \\ \text{for change of variables} \\ \text{is } r\end{array}\right]$$

$$= 3\int_0^{2\pi}\int_1^3 r\left[3 - r\cos\theta\right] dr\, d\theta$$

$$= 3\int_0^{2\pi}\int_1^3 (3r - r^2\cos\theta)\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{3r^2}{2} - \frac{r^3}{3}\cos\theta\right]_1^3 d\theta$$

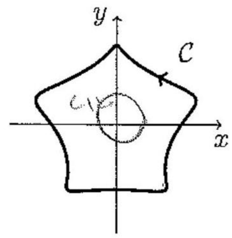
$$= 3\int_0^{2\pi}\left(\frac{27}{2} - 9\cos\theta - \frac{3}{2} + \frac{\cos\theta}{3}\right) d\theta$$

$$= 3\int_0^{2\pi}\left(12 - \frac{26\cos\theta}{3}\right) d\theta$$

$$= 3\left[12\cdot 2\pi - \frac{26}{3}\left[\sin\theta\right]_0^{2\pi}\right]$$

$$= 72\pi$$

7. (12 points) Let C be the curve



Find $\oint_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \cos(x^3) + ye^{xy},\ \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle.$$

(*Hint: Try writing* $\mathbf{F}$ *as a sum of two vector fields that we know how to integrate around* C.)

Given $\vec{F} = \left\langle \frac{-y}{x^2+y^2} + \cos(x^3) + ye^{xy},\ \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle$

Let $\vec{F} = \vec{F_1} + \vec{F_2}$, where $\vec{F_1} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

$\vec{F_2} = \left\langle \cos(x^3) + ye^{xy},\ e^{e^y} + xe^{xy} \right\rangle$

Let C_1 be a circle of Radius R, oriented counterclockwise such the C_1 lies completely inside D, where D is the region whose boundary curve is C.

Parametrize C_1 by $\vec{r}(t) = \langle R\cos t, R\sin t \rangle$ $0 \le t < 2\pi$

$\vec{r}'(t) = \langle -R\sin t, R\cos t \rangle$

$$\oint_{C_1} \vec{F_1} \cdot d\vec{r} = \int_0^{2\pi} \left\langle -\frac{R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t \rangle\, dt$$

$$= \int_0^{2\pi} dt = 2\pi.$$

Consider $\text{curl}_z \vec{F_1} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right)$

$$= x \cdot \frac{-2x}{(x^2+y^2)^2} + \frac{1}{x^2+y^2} - \frac{y \cdot 2y}{(x^2+y^2)^2} + \frac{1}{x^2+y^2}$$

$$= \frac{-2x^2 - 2y^2 + 2x^2 + 2y^2}{(x^2+y^2)^2} = 0 \quad (x,y) \neq (0,0)$$

$\therefore \text{curl}_z \vec{F_1} = 0 \quad \forall (x,y) \neq (0,0)$

Consider D_1 to be the region outside circle C_1, inside boundary curve C.

$\therefore$ By Green's thm, $\iint_{D_1} \text{curl}_z \vec{F_1}\, dA = \oint_C \vec{F_1} \cdot \vec{dr} - \oint_{C_1} \vec{F_1} \cdot \vec{dr}$

$0 \quad [\because (0,0) \notin D_1] \quad = \oint_C \vec{F_1} \cdot \vec{dr} - 2\pi.$

$\therefore \boxed{\oint_C \vec{F_1} \cdot \vec{dr} = 2\pi}$

Consider $\text{curl}_z \vec{F_2} = \frac{\partial}{\partial x}\left[e^{xy} + xe^{xy}\right] - \frac{\partial}{\partial y}\left[ye^{xy}\right]$

$= x \cdot e^{xy} \cdot y + e^{xy} - y \cdot e^{xy} \cdot x - e^{xy}$

$= (xy+1)e^{xy} - (xy+1)e^{xy}$

$= 0 \quad \forall (x,y) \in D$

$\therefore$ By Green's thm. $\oint_C \vec{F_2} \cdot \vec{dr} = \iint_D \text{curl}_z \vec{F_2}\, dA$

$\therefore \boxed{\oint_C \vec{F_2} \cdot \vec{dr} = 0} \quad [\because \text{curl}_z \vec{F_2} = 0]$

$\because \vec{F} = \vec{F_1} + \vec{F_2},$

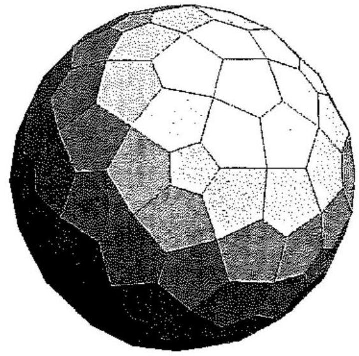
$\oint_C \vec{F} \cdot \vec{dr} = \oint_C \vec{F_1} \cdot \vec{dr} + \oint_C \vec{F_2} \cdot \vec{dr}$

$= 2\pi + 0$

$= 2\pi$

$\therefore \boxed{\oint_C \vec{F} \cdot \vec{dr} = 2\pi}$

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{a}_j$ so that $\mathbf{a}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{a}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

a) $\because \; S = S_1 \cup S_2 \cup \cdots \cup S_k$

$$\iint_S \vec{F} \cdot \vec{dS} = \sum_{j=1}^{k} \iint_{S_j} \vec{F} \cdot \vec{dS_j}$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \iint_{S_j} \vec{N_i} \cdot \vec{dS_j} \qquad [\because \vec{F} = \vec{N_1} + \vec{N_2} + \cdots + \vec{N_k}]$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \iint_{S_j} \vec{N_i} \cdot \vec{a_j}\, dS_j \qquad [\because a_j \text{ is outward unit normal}]$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \vec{N_i} \cdot \vec{a_j} \iint_{S_j} dS_j \qquad [\because \vec{N_i} \cdot \vec{a_j} \text{ is constant for given surface } S_j]$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \vec{N_i} \cdot \vec{a_j}\, \text{Area}(S_j)$$

$$= \sum_{j=1}^{k} \sum_{i=1}^{k} \vec{N_i} \cdot \vec{N_j} \quad [\because \vec{N_j} = \text{Area}(S_j)\,\vec{a_j}]$$

$$= (\vec{N_1} + \vec{N_2} + \dots + \vec{N_k}) \cdot (\vec{N_1} + \vec{N_2} + \dots + \vec{N_k})$$

$$= \vec{F} \cdot \vec{F}$$

$$= \|\vec{F}\|^2$$

Hence, proved.

b) $\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div} \vec{F} \, dV$ [By divergence thm]

$= 0$ [$\because \operatorname{div} \vec{F} = 0$ explained below)

$\therefore \|\vec{F}\|^2 = 0$

$\Rightarrow \boxed{\vec{F} = \vec{0}}$

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \nabla \cdot \vec{N_1} + \nabla \cdot \vec{N_2} + \nabla \cdot \vec{N_3} + \dots + \nabla \cdot \vec{N_k}$$

$\left[\because \vec{N_i} \text{ is a constant vector, } \nabla \cdot \vec{N_i} = \frac{\partial x_i}{\partial x} + \frac{\partial y_i}{\partial y} + \frac{\partial z_i}{\partial z} = 0 \right]$

$$= 0 + 0 + \dots + 0$$

$$= 0$$