Math 32B Final Clockwise

ASHWIN RANADE

TOTAL POINTS

61 / 90

QUESTION 1

1 Fubini's Theorem 5 / 6

+ **6 pts** Correct answer (2/3)(e^27 - 1).
✓ + **2 pts** **(Partial credit) New x limits are 0 to 9.**
✓ + **2 pts** **(Partial credit) New y limits are 0 to sqrt(x).**
✓ + **1 pts** **(Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).**
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 9, sqrt(x) <= y <= 3

QUESTION 2

2 Stokes' Theorem 8 / 8

✓ + **8 pts** **Correct answer 248.**
+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 208.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stokes theorem.

QUESTION 3

3 Line integral 4 / 12

+ **4 pts** Correct parametrization
✓ + **2 pts** **Partial crerdits for parametrization**
+ **4 pts** Correct integral formula
✓ + **2 pts** **Partial credits for integral**
+ **4 pts** Correct calculation and final answer
+ **2 pts** Partial credits for calculation
+ **1 pts** Almost makes no sense
+ **0 pts** Nothing correct
- **1 pts** Tiny calculation error

QUESTION 4

4 Moment of inertia 11 / 14

✓ + **1 pts** **a) Correct limits $$0\leq\rho\leq\frac1{10}$$**
✓ + **1 pts** **a) Correct limits $$0\leq\theta<2\pi$$**
+ **1 pts** a) Correct upper bound $$\phi\leq \pi$$
+ **2 pts** a) Correct lower bound $$\phi\geq \frac{2\pi}3$$
✓ + **1 pts** **b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**
✓ + **1 pts** **b) Correct integrand $$3(x^2+y^2)$$ (must substitute $$\delta=3$$ into formula from formula sheet to gain credit)**
✓ + **2 pts** **b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**
✓ + **1 pts** **b) Correctly simplifying $$3(x^2+y^2)$$ to $$3\rho^2\sin^2\phi$$**
✓ + **2 pts** **b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**
+ **1 pts** b) Correct answer of $$\frac{\pi}{400000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)
✓ + **1 pts** **Solution thoroughly explained, using full sentences**
✓ + **1 pts** **Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)**
+ **0 pts** No credit due

QUESTION 5

5 Probability 11 / 14

✓ + **2 pts** Correct limits (max 4 pts)

+ **1 pts** Correct limits

+ **1 pts** Correct limits

✓ + **2 pts** Correct integrand (max 5 pts)

✓ + **2 pts** correct integrand

✓ + **1 pts** Correct integrand

✓ + **2 pts** Computations (max 5 pts)

✓ + **2 pts** Computations

+ **1 pts** Computations

+ **0 pts** No credit due

QUESTION 6

6 Divergence Theorem 14 / 14

✓ + **4 pts** Correct divergence

✓ + **7 pts** Correct parametrization of $$ \mathcal{W}$$

✓ + **3 pts** Correct evaluation of triple integral

+ **2 pts** Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])

+ **0 pts** No credit

QUESTION 7

7 Vector line integral 4 / 12

✓ + **4 pts** Write F as a sum of vortex field and a conservative field

+ **2 pts** Vortex field has integral 2pi over this C

+ **2 pts** Compute curl_z F_2 or show F_2 is conservative

+ **3 pts** Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0

+ **1 pts** Arrive at correct answer, 2pi, by valid method

+ **0 pts** Incorrect

+ **2 pts** Mostly correct argument that integral of F_2 is 0

+ **1 pts** curl_z F_2 minor error

Can't apply Green's thm directly to F because of singularity

QUESTION 8

8 Surface integral 4 / 10

+ **3 pts** Decompose flux integral

+ **1 pts** Partial credit for decomposition

+ **2 pts** Do component integrals

✓ + **1 pts** Partial credit for component integrals

+ **1 pts** Combine integrals

✓ + **2 pts** Used divergence theorem (part (b))

+ **1 pts** Correct (and justified) div(F) (part (b))

✓ + **1 pts** Clear and well-explained solution

+ **0 pts** No credit due

||F||≠Area(S); that would only be true if the b_j were parallel.

||F|| is a scalar, and cannot be part of a dot product, and the same goes for the scalar surface integral ∬1dS.

W need not be symmetric; div(F)=0 because F is constant.

Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: Ashwin Ranade
SID: 805152956
TA Section: 3F

Time Limit: 180 Minutes

Version 0

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\, dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\, dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\, dA$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\, dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\, dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\, dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x,y,z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x,y,z)\, dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\,\delta(x,y,z)\, dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\,\delta(x,y,z)\, dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\,\delta(x,y,z)\, dV$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2+z^2)\,\delta(x,y,z)\, dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2+z^2)\,\delta(x,y,z)\, dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2+y^2)\,\delta(x,y,z)\, dV$.

Probability formulas

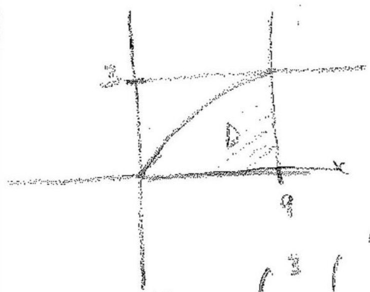
- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x, y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x, y)\,dxdy = 1$
  - The probability that $(X, Y) \in \mathcal{D}$ is $\mathbb{P}[(X, Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x, y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X, Y)$ is $\mathbb{E}[f(X, Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,p_{X,Y}(x, y)\,dxdy$.

1. (6 points) Find $\int_0^3 \int_{y^2}^9 e^{x^{\frac{3}{2}}}\,dxdy$.

Use Fubini's Thereom. First, draw the region.

$0 \leq y \leq 3$
$y^2 \leq x \leq 9$

$x = y^2$
$y = \pm\sqrt{x}$ Since $y \geq 0$, we only look @ the positive square root.



(Fubini)

$$\int_0^3 \int_{y^2}^9 e^{x^{\frac{3}{2}}}\,dx\,dy = \int_0^9 \int_0^{\sqrt{x}} e^{x^{\frac{3}{2}}}\,dy\,dx$$

$$= \int_0^9 e^{x^{3/2}}\, y \Big|_{y=0}^{y=\sqrt{x}}\,dx = \int_0^9 e^{x^{3/2}}\sqrt{x}\,dx$$

Use u-sub. $u = x^{3/2}$ $du = \frac{3}{2}x^{\frac{1}{2}} = \frac{3}{2}\sqrt{x}\,dx$

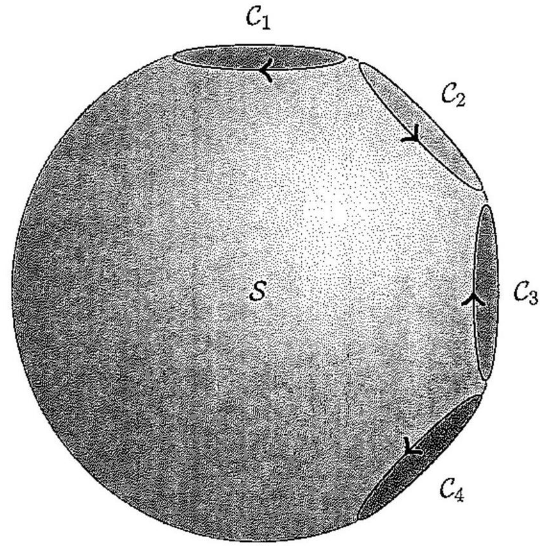
$\sqrt{x}\,dx = \frac{2}{3}du$

$$\int_0^9 e^{x^{3/2}}\sqrt{x}\,dx = \frac{2}{3}\int_{u(0)}^{u(9)} e^u\,du = \frac{2}{3}e^u\Big|_{u(0)}^{u(9)}$$

Compute bounds.
$u(9) = (9)^{3/2} = (\sqrt{9})^3 = 3^3 = 27$
$u(0) = (0)^{3/2} = 0$

Hence, $\frac{2}{3}e^u\Big|_0^{27}$ $\boxed{= \frac{2}{3}e^{27}}$

2. (8 points) Let $\mathcal{S}$ be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_{S} \operatorname{curl}\mathbf{F}\cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F}\cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F}\cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r}$.

Use Stoke's theorem. $\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

The boundary of the sphere is the union of the 4 curves.

If we are walking along C, we need our left hand to be over the surface so we have proper orientation.

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r} = \oint_{C_1} \vec{F}\cdot d\vec{r} - \oint_{C_2} \vec{F}\cdot d\vec{r} + \oint_{C_3} \vec{F}\cdot d\vec{r} - \oint_{C_4} \vec{F}\cdot d\vec{r}$$

Plug in and solve.

$$20 = \oint_{C_1} \vec{F}\cdot d\vec{r} - 305 + 104 - 27$$

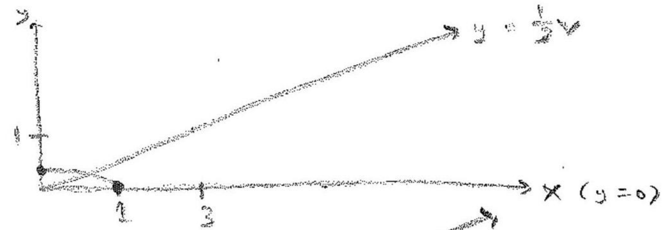
$$\boxed{\oint_{C_1} \vec{F}\cdot d\vec{r} = 20 + 305 - 104 + 27}$$

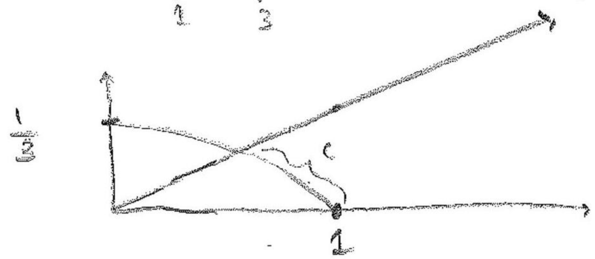
3. (12 points) Let C be the part of the ellipse $x^2+9y^2=1$ bewteen $y=0$ and $y=\frac{1}{3}x$ in the first quadrant. Find $\int_C x\sqrt{\frac{1}{9}x^2+9y^2}\,ds$.

Draw the region.

$x^2+9y^2=1$
$x^2+(0)=1$
$0^2+9y^2=1 \quad y^2=1/9$
$y=\pm\frac{1}{3}$

} finding shape of ellipse





$\int_C F\,ds = \int_a^b \vec{F}(\vec{r}(t))\,\|\vec{r}\,'(t)\|\,dt$

Param the curve. For a circle, $x^2+y^2=1 \to \langle \cos\theta \;\; \sin\theta \rangle$

So, $x^2+9y^2=1$, so $\quad \vec{r}(t) = \frac{1}{3}\langle 3\cos\theta,\ \frac{1}{3}\sin\theta \rangle$

check: $\vec{r}(0) = \frac{1}{3}\langle 3, 0\rangle = \langle 1, 0\rangle$ ✓

$\vec{r}\,'(t) = \frac{1}{3}\langle -3\sin\theta,\ \frac{1}{3}\cos\theta \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{\frac{1}{9}\left(9\sin^2\theta + \frac{1}{9}\cos^2\theta\right)} \quad d\theta$

$\int_C F\,ds = \int_a^b \cos\theta\ \sqrt{\frac{1}{9}\cancel{\left(\frac{9\cos^2\theta}{9}+\frac{9}{9}\sin^2\theta\right)}}\ \ \frac{1}{3}\sqrt{9\sin^2\theta+\frac{1}{9}\cos^2\theta}\ d\theta$

$\int_C F\,ds = \int_a^b \frac{1}{9}\cos\theta\ \sqrt{9\sin^2\theta+\frac{1}{9}\cos^2\theta}\ d\theta$

we know $x^2+9y^2=1 \quad \frac{1}{3}(9\cos^2\theta+\sin^2\theta)=1$

$9\cos^2\theta+\sin^2\theta = 3$

$\sin^2\theta = 3-9\cos^2\theta$

$\int_C F\,ds = \frac{1}{9}\int_a^b \cos\theta\ \sqrt{9(3-9\cos^2\theta)+\frac{1}{9}\cos^2\theta}\ d\theta$

$$\int_C F\,ds = \frac{1}{9}\int_a^b \cos\theta \sqrt{27 - 81\cos^2\theta + \frac{9}{81}\cos^2\theta}\; d\theta$$

$$\int_C F\,ds = \frac{1}{9}\int_a^b \cos\theta \sqrt{27 + \left(\frac{-81^2}{81} + \frac{9}{81}\right)\cos^2\theta}\; d\theta$$

$$\int_C F\,ds = \frac{1}{9}\int_a^b \cos\theta \sqrt{\frac{81}{3} - \frac{(9-81^2)\cos^2\theta}{81}}\; d\theta$$

$$= \frac{1}{9}\int_a^b \cos\theta \sqrt{\frac{27(81) + (81^2-9)\cos^2\theta}{81}}\; d\theta$$

Find bounds $\quad \frac{1}{9}\sin\theta = \frac{1}{3}\cos\theta$ } line

$\cos^2\theta + \sin^2\theta = 1$

$\frac{1}{9}\sin^2\theta = \frac{1}{3}\cos\theta\sin\theta$ $\qquad y = \frac{1}{3}\sqrt{2}$

$x^2 + 9(1/3x)^2 = 1$

$x^2 + \frac{9x^2}{9} = 1 \qquad 2x^2 = 1 \qquad x = \pm\sqrt{2}$

$x = r\cos\theta \qquad \theta = \tan^{-1}(y/x)$

$y = r\sin\theta \qquad \theta = \tan^{-1}(1/3)$

$$= \frac{1}{81}\int_{\tan^{-1}(1/3)}^{\pi/2} \cos\theta \sqrt{27(81) + (81^2-9)\cos^2\theta}\; d\theta$$

$u = \cos^2\theta \qquad du = 2\cos\theta\, d\theta$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{81}\right)\int_{u(\tan^{-1}(1/3))}^{u(\pi/2)} \sqrt{u(81^2-9) + 27(81)}\; d\theta$$

antiderivative $\quad \frac{2}{3}\,\frac{u^{3/2}}{(81^2-9)}$

$\cos^2(\pi/2) = 1$

$$\boxed{\int_C F\,ds = \left(\frac{1}{2}\right)\left(\frac{1}{81}\right)\left(\frac{2}{3}\right)\frac{1}{(81^2-9)}\left(\sqrt{(81^2-9)(1)+27(81)} - \sqrt{\cos^2(\tan^{-1}(1/3))(81^2-9) + 27(81)}\right)}$$

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2+y^2+z^2 \le \frac{1}{100}$ and $\sqrt{3}z \le -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x,y,z) = 3\,\text{kg}\,\text{m}^{-3}$.
   (a) Write $\mathcal{W}$ using spherical coordinates.
   (b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*

a)

$(x,y,z) \to (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)$

$x^2+y^2+z^2 \le \frac{1}{100}$ $\sqrt{3}z \le -\sqrt{x^2+y^2}$ Plug in spherical conversions.

$\rho^2 \le \frac{1}{100}$ $\sqrt{3}\rho\cos\phi \le -\sqrt{\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta}$

$\rho \le \frac{1}{10}$ $\sqrt{3}\rho\cos\phi \le -\sqrt{\rho^2\sin^2\phi\,(\cancel{\cos^2\theta+\sin^2\theta})}$

$\rho \le \frac{1}{10}$ $\sqrt{3}\cancel{\rho}\cos\phi \le -\cancel{\rho}\sin\phi$

$-\sqrt{3} \le \tan\phi$

$\tan\phi \ge -\sqrt{3}$ at $\frac{\pi}{6}$, $\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$ find intersection

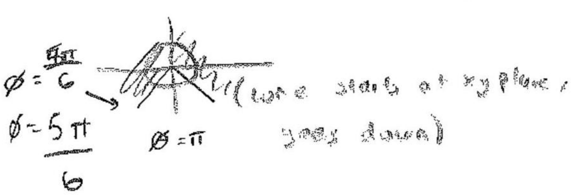
$\tan(\pi/6) = \sqrt{3}$

$\tan\left(\frac{5\pi}{6}\right)$ and $\tan\left(\frac{-\pi}{6}\right) = -\sqrt{3}$

sphere w/ an upside down cone

$$\boxed{\begin{aligned} &\text{We let } 0 \le \rho \le \tfrac{1}{10} \\ &0 \le \theta \le 2\pi \\ &0 \le \phi \le \tfrac{5\pi}{6} \end{aligned}}$$

(we want the area above the cone)



b) Find $I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dV$
convert to spherical

$x^2+y^2 = \rho^2\sin\phi$

$$I_z = 3\int_0^{2\pi}\int_0^{5\pi/6}\int_0^{1/10} \rho^2\sin\phi\,\overbrace{\rho^2\sin\phi}^{\text{Jacobian}}\,dV$$

$$I_z = 3\int_0^{2\pi}\int_0^{5\pi/6} \sin^2\phi \left(\frac{\rho^5}{5}\right)\Big|_0^{1/10} d\rho\, d\phi\, d\theta$$

$$I_z = \left(\frac{3}{5}\right)\left(\frac{1}{10^5}\right)\int_0^{2\pi}\int_0^{5\pi/6} \sin^2\phi\, d\phi\, d\theta$$

$\sin(2x) = 2\sin(x)\cos(x)$

$\cos(2x) = 1 - 2\sin^2(x)$

$2\sin^2(x) = 1 - \cos(2x)$

$\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) = \frac{1}{2}(1 - \cos(2x))$

$$I_z = \left(\frac{1}{2}\right)\left(\frac{3}{5}\right)\left(\frac{1}{10^5}\right)\int_0^{2\pi}\int_0^{5\pi/6} 1 - \cos(2x)\, d\phi\, d\theta$$

$$I_z = \frac{3}{10^6}\int_0^{2}\left(\phi - \frac{1}{2}\sin 2\phi\right)\Big|_0^{5\pi/6} d\theta = \frac{3}{10^6}\int_0^{2\pi} \frac{5\pi}{6} - \frac{5\pi}{3}\sin\left(\frac{10\pi}{6}\right) d\theta$$

$$I_z = \frac{3}{10^6}\int_0^{2\pi} \frac{5\pi}{6} - \frac{5\pi}{3}\left(\underbrace{\sin\left(\frac{5}{3}\pi\right)}_{-\frac{1}{2}}\right) d\theta \qquad \sin(\pi/3) = \frac{1}{2}$$

$$I_z = \frac{3}{10^6}\int_0^{2\pi} \frac{5\pi}{6} + \frac{5\pi}{6}\, d\theta$$

$$I_z = \frac{3}{10^6}\int_0^{2\pi} \frac{10\pi}{6}\, d\theta \boxed{= \frac{3}{10^6}\left(\frac{10\pi}{6}\right)(2\pi)\ kgm^2}$$

units: $\iiint \frac{(m^2)\,kg}{m^3} = m^2 kg$

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \frac{1}{\sqrt{3}}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{3}{175}\dfrac{xy^2}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.

$$E\left[f(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p(x,y)\,dx\,dy$$

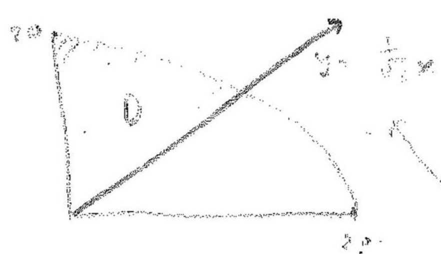
$$E\left[\sqrt{X^2+Y^2}\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2}\,p(x,y)\,dA$$

However, since $p_{X,Y}(x,y) = 0$ outside D,

$$E\left[\sqrt{X^2+Y^2}\right] = \iint_D \sqrt{x^2+y^2}\,\frac{3}{175}\,\frac{xy^2}{(x^2+y^2)^{3/2}}\,dA$$

$$E = \frac{3}{175}\iint_D \frac{1}{x^2+y^2}\,xy^2\,dA$$

Draw region.



Convert to polar.

$x = r\cos\theta$

$y = r\sin\theta$

becomes $\cancel{r}\sin\theta = \frac{\cancel{r}\cos\theta}{\sqrt{3}}$ $\quad\tan\theta = \frac{1}{\sqrt{3}}$ $\quad \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$

$\theta \geq \frac{\pi}{3}$ $\quad \theta \leq \frac{\pi}{2}$

Jacobian

$$\text{So, } E = \int_{\pi/3}^{\pi/2}\int_0^{20}\left(\frac{3}{175}\right)\frac{\cancel{r^2}\sin^2\theta\, r\cos\theta\, r}{\cancel{r^2}}\,dr\,d\theta$$

$$E = \frac{3}{175}\int_{\pi/3}^{\pi/2}\sin^2\theta\cos\theta\int_0^{20} r^2\,dr\,d\theta = \frac{\cancel{3}}{175}\int_{\pi/3}^{\pi/2}\sin^2\theta\cos\theta\,\frac{r^3}{\cancel{3}}\Big|_0^{20}\,d\theta$$

$$E = \frac{1}{175}\int_{\pi/3}^{\pi/2}\sin^2\theta\cos\theta\;20^3\,d\theta$$

$$E = \frac{20^3}{175}\int_{\pi/3}^{\pi/2} \sin^2\theta\cos\theta \, d\theta \qquad \begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \end{array}$$

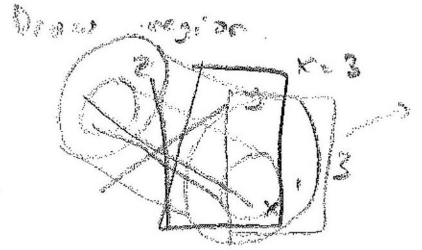
$$E = \frac{20^3}{175}\int_{u(\pi/3)}^{u(\pi/2)} u^2 \, du = \frac{20^3}{175} \left. \frac{u^3}{3} \right|_{u(\pi/3)}^{u(\pi/2)}$$

$\sin(\pi/2) = 1$ $\quad \sin(\pi/3) = \frac{1}{2}$

$$\boxed{E = \frac{20^3}{3(175)}\left(\frac{1^3}{3} - \frac{(\frac{1}{2})^3}{3}\right)}$$

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $y^2+z^2=1$, $y^2+z^2=9$ and the planes $x=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F}=\left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 2z \right\rangle$ across $\mathcal{S}$.

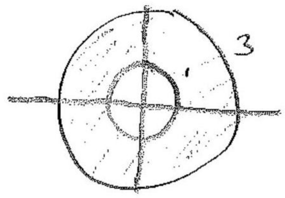
Draw region



$x^2+y^2=9$

$z=3$ $\quad z=x$

Project on yz plane



$x=3,\ x=y \qquad y \le x \le 3$ (the max y will be is 3)

$\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\,dV$

$\iint_S \operatorname{curl}\vec{F}\cdot d\vec{S} = \oint_C \vec{F}\cdot d\vec{r}$

Find a potential function for $\vec{F}$.

$\frac{\partial F}{\partial x} = y\,(x^2+y^2)^{-1} \qquad \int x^{-2} = -x^{-1}$

$\int x^{-1}\,dx = \ln|x|$

$F = \int \frac{\partial F}{\partial x}\,\partial x = \frac{y\ln|x^2+y^2|}{2x}$

$\frac{\partial F}{\partial y} = \frac{\ln|x^2+y^2|}{2x}$ $\qquad$ Try divergence instead.

$\operatorname{div}(\vec{F}) = \cancel{\frac{y(2x)}{(x^2+y^2)^2}} - \cancel{\frac{x(2y)}{(x^2+y^2)^2}} + 2 = 2$

$\operatorname{div}(\vec{F}) = 2$

Use divergence thereom.

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \operatorname{div}(\vec{F})\, dV = 2 \iiint_W dV$$

Convert to cylindrical coordinates. Rotate volume so cylinder points along z axis, because $\vec{F} = 2$ = constant.

$(x, y, z) \rightarrow (r\cos\theta, r\sin\theta, z)$

$$2\iiint_W dV = 2\int_0^{2\pi}\int_1^3\int_{r\cos\theta}^{3} \overset{\text{Jacobian}}{r}\, dz\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_1^3 r\,(3 - r\cos\theta)\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_1^3 3r - r^2\cos\theta\, dr\, d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{3r^2}{2} - \frac{r^3\cos\theta}{3}\right)\Big|_1^3\, d\theta$$

$$= 2\int_0^{2\pi}\left(\frac{27}{2} - \frac{27\cos\theta}{3}\right) + \left(-\frac{3}{2} + \frac{\cos\theta}{3}\right) d\theta$$

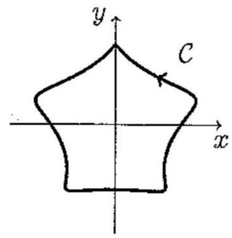
$$= 2\int_0^{2\pi} \frac{24}{2} - \frac{26\cos\theta}{3}\, d\theta$$

$$= 2(2\pi)(12) - \frac{26}{3}\int_0^{2\pi}\cos\theta\, d\theta$$

$$-\frac{26}{3}\cancel{\sin\theta\Big|_0^{2\pi}} \rightarrow 0$$

$$\boxed{= 48\pi}$$

7. (12 points) Let $\mathcal{C}$ be the curve



Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$\frac{d}{dx} x^{-1} = -1\, x^{-2}$

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \sin(x^5) + 2ye^{2xy},\ \frac{x}{x^2+y^2} + e^{\cos(y)} + 2xe^{2xy} \right\rangle.$$

(*Hint: Try writing* $\mathbf{F}$ *as a sum of two vector fields that we know how to integrate around* $\mathcal{C}$.)

$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \text{curl}_z \vec{F}\ \partial A$

$\vec{F}(x,y) = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle + \left\langle \sin(x^5) + 2ye^{2xy}, 2xe^{2xy} \right\rangle$

If $\vec{F}$ is conservative, $\oint_C \vec{F}\cdot d\vec{r} = 0$

If $\text{curl}_z \vec{F} = 0$, $\oint_C \vec{F}\cdot d\vec{r} = 0$, by Greene's Thereom.

$\text{curl}_z \vec{F} = \frac{\partial \vec{F_2}}{\partial x} - \frac{\partial \vec{F_1}}{\partial y}$

$\vec{F} = \frac{1}{2}(x^2+y^2)^{-1}$

$\frac{\partial \vec{F}}{\partial x} = \frac{-2x}{(x^2+y^2)^{-2}}$ $\quad \frac{\partial \vec{F}}{\partial y} = \frac{-y}{(x^2+y^2)^{-2}}$

$\vec{F} = \frac{1}{x^2+y^2}\langle -y, x\rangle + \langle \sin(x^5), e^{\cos(y)}\rangle + 2e^{2xy}\langle y, x\rangle$

$\text{curl}_z \vec{F} = \frac{(x^2+y^2)+(2x)(x)}{(x^2+y^2)^2} + 0 + \cancel{2e^{2xy}} + \cancel{2e^{2xy}(2y)} \Big\} \frac{\partial F_2}{\partial x}$

$- \left(\frac{(-1)(x^2+y^2) + (-y)(2y)}{(x^2+y^2)^2} + 0 + \cancel{2e^{2xy}} + \cancel{2e^{2xy}(2x)} \right) \Big\} \frac{\partial F_1}{\partial y}$

$\text{curl}_z \vec{F} = \frac{2(x^2+y^2) + 2(x^2+y^2)}{(x^2+y^2)^2} = \frac{4(x^2+y^2)}{(x^2+y^2)^2} = \frac{4}{x^2+y^2}$

$\cancel{\text{curl}_z \vec{F} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}$

So $\oint_C \vec{F} \cdot d\vec{r} = \iint_D \frac{4}{x^2+y^2} \, dA$ (Green's theorem)

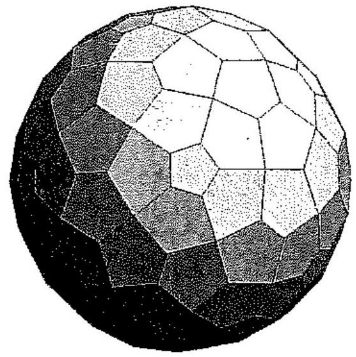
converting to polar

$$\oint \vec{F} \cdot d\vec{r} = \iint_{D_0} \frac{4}{r^2} (r) \, dA = \iint_{D_0} \frac{4}{r} \, dA$$

The region is symmetric around the origin.
Our function $\frac{4}{r}$ is also symmetric around the origin.
Hence, everything cancels, and

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_{D_0} \frac{4}{r} \, dA \quad \underline{= 0}$$

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{b}_j$ so that $\mathbf{b}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{b}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

a) $\vec{F} = \sum_{1}^{k} N_k = \text{Area}(S_1)\,\hat{b}_1 + \ldots + \text{Area}(S_k)\,\hat{b}_k$

$\|\vec{F}\| = \text{Area}(S_1) + \text{Area}(S_2) + \ldots + \text{Area}(S_k)$, since $\|\hat{b}_j\| = 1$

$\|\vec{F}\| = \text{Area}(S) = \iint_S 1\, dS$ (unit normal)

since $S = S_1 \cup S_2 \cup \ldots \cup S_k$

Recall $\|\vec{F}\|^2 = \vec{F} \cdot \vec{F} = \|\vec{F}\|\,\|\vec{F}\|$

$\|\vec{F}\| \cdot \vec{F} = \vec{F} \cdot \left(\iint_S 1\, dS\right)$ dot product both sides by $\vec{F}$

$\|\vec{F}\|^2 = \iint_S \vec{F} \cdot d\vec{S}$ (since $\hat{n} = \hat{b}$ = unit outward normal)

b) Show that $\vec{F} = 0$

$\|\vec{F}\|^2 = \iint_S \vec{F} \cdot d\vec{S}$

Use stoke's Thereom/Divergence Thereom.

$\|\vec{F}\|^2 = \iint_S \vec{F} \cdot d\vec{S}$ $\qquad \iint_S \text{curl } \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$

$$\iiint_W \text{div } \vec{F} \, dV = \iint_S \vec{F} \cdot \hat{n} \, dS$$

$\vec{F} = \vec{N}_1 + \vec{N}_2 + \dots + \vec{N}_k \qquad \vec{N}_j = \text{Area}(S_j)\, \hat{b}_j$

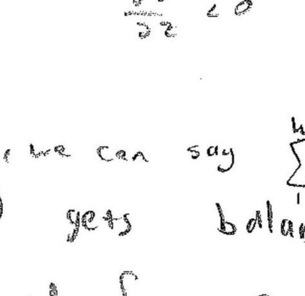
If we show $\text{curl}(\vec{G}) = \vec{F}$, then we can prove $\iint_S \vec{F} \cdot d\vec{S} = 0$, since S is closed. $\vec{G}$ is an arbitrary function.

We can also show $\text{div } \vec{F} = 0$ and then show $\iint_S \vec{F} \cdot d\vec{S}$ using divergence thereom.

$\vec{F} = \sum \text{Area}(S_j)\, \hat{b}_j$

$$\text{div } \vec{F} = \sum_{1}^{K} \text{div}\left(\text{Area}(S_k)\, \hat{b}_k\right) \qquad \text{div}\left(\text{Area}(S_k)\, \hat{b}_k\right) = \text{Area}(S_k) \frac{\partial b}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial b}{\partial z}$$

$\hat{b}$ is outward pointing. It surrounds the whole surface.



these cancel out on both poles. Also, on both poles $\frac{\partial b}{\partial x} + \frac{\partial b}{\partial y} = 0$, since $\hat{b}$ only points in the z direction.

By symmetry, we can say $\sum_{1}^{k} \text{div}(b_k) = 0$, because the $\text{div}(\vec{b})$ gets balanced out by opposite side of surface.

Hence, $\sum_{1}^{k} \text{div}(\vec{b}_k) = 0$, so $\text{div}(\vec{F}) = 0$.

So, by divergence thereom,

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_W \text{div}(\vec{F})\, dV = \iiint_W 0\, dV \boxed{= 0}$