Math 32B Final Counterclockwise

[redacted]

TOTAL POINTS

75 / 90

QUESTION 1

1 Fubini's Theorem 6 / 6

✓ **+ 6 pts Correct answer (2/3)(e^8 - 1).**

+ **2 pts** (Partial credit) New x limits are 0 to 4.
+ **2 pts** (Partial credit) New y limits are 0 to sqrt(x).
+ **1 pts** (Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 4, sqrt(x) <= y <= 2

QUESTION 2

2 Stokes' Theorem 8 / 8

✓ **+ 8 pts Correct answer 402.**

+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 362.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stoke's theorem.

QUESTION 3

3 Line integral 6 / 12

+ **4 pts** Correct parametrization

✓ **+ 2 pts Partial credits for parametrization**

✓ **+ 4 pts Correct integral formula**

+ **2 pts** Partial credits for integral
+ **4 pts** Correct calculation
+ **2 pts** Partial credits for calculation
+ **1 pts** Almost makes no sense.
+ **0 pts** Nothing correct

- **1 pts** Tiny calculation error

QUESTION 4

4 Moment of inertia 11 / 14

✓ **+ 1 pts a) Correct limits $$0\leq\rho\leq\frac1{10}$$**

✓ **+ 1 pts a) Correct limits $$0\leq\theta<2\pi$$**

✓ **+ 1 pts a) Correct upper bound $$\phi\leq \pi$$**

+ **2 pts** a) Correct lower bound $$\phi\geq \frac{2\pi}3$$

✓ **+ 1 pts b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**

✓ **+ 1 pts b) Correct integrand $$5(x^2+y^2)$$ (must substitute $$\delta=5$$ into formula from formula sheet to gain credit)**

✓ **+ 2 pts b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correctly simplifying $$5(x^2+y^2)$$ to $$5\rho^2\sin^2\phi$$**

✓ **+ 2 pts b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**

+ **1 pts** b) Correct answer of $$\frac{\pi}{240000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)

✓ **+ 1 pts Solution thoroughly explained, using full sentences**

+ **1 pts** Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)
+ **0 pts** No credit due

QUESTION 5

5 Probability 14 / 14

✓ + **14 pts** **Full points**

+ **0 pts** No points

+ **2 pts** Correctly labeled region (all or nothing)

+ **3 pts** Correctly set-up integral (max 6 pts)

+ **2 pts** Correctly set-up integral

+ **1 pts** Correctly set-up integral

+ **3 pts** Evaluation of integral (max 6 pts)

+ **2 pts** Evaluation of integral

+ **1 pts** Evaluation of integral

QUESTION 6

6 Divergence Theorem 11 / 14

✓ + **4 pts** **Correct divergence**

✓ + **7 pts** **Correct parametrization of $$ \mathcal{W}$$**

+ **3 pts** Correct evaluation of correct triple integral (implicit in the grading process was that this rubric item meant that you could have also correctly computed the volume using high school geometry)

+ **2 pts** Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])

+ **0 pts** No credit

QUESTION 7

7 Vector line integral 12 / 12

✓ + **4 pts** **Write F as a sum of vortex field and a conservative field**

✓ + **2 pts** **Vortex field has integral 2pi over this C**

✓ + **2 pts** **Compute curl_z F_2 or show F_2 is conservative**

✓ + **3 pts** **Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0**

✓ + **1 pts** **Arrive at correct answer, 2pi, by valid method**

+ **0 pts** Incorrect

+ **2 pts** Mostly correct argument that integral of F_2 is 0

+ **1 pts** curl_z F_2 minor error

QUESTION 8

8 Surface integral 7 / 10

✓ + **3 pts** **Decompose flux integral**

+ **1 pts** Partial credit for decomposition

+ **2 pts** Do component integrals

+ **1 pts** Partial credit for component integrals

+ **1 pts** Combine integrals

✓ + **2 pts** **Used divergence theorem (part (b))**

✓ + **1 pts** **Correct (and justified) div(F) (part (b))**

✓ + **1 pts** **Clear and well-explained solution**

+ **0 pts** No credit due

Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: [redacted]
SID:
TA Section:

Time Limit: 180 Minutes

Version 0

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

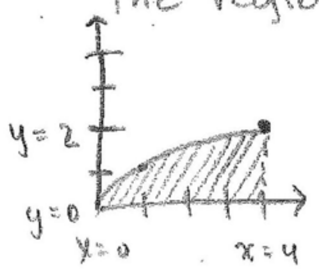
- If $\mathcal{D}$ is a lamina with mass density $\delta(x,y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x,y)\,dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\,\delta(x,y)\,dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\,\delta(x,y)\,dA$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\,\delta(x,y)\,dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\,\delta(x,y)\,dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2+y^2)\,\delta(x,y)\,dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x,y,z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x,y,z)\,dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\,\delta(x,y,z)\,dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\,\delta(x,y,z)\,dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\,\delta(x,y,z)\,dV$.
  - The center of mass is $(x_{\mathrm{CM}}, y_{\mathrm{CM}}, z_{\mathrm{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2+z^2)\,\delta(x,y,z)\,dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2+z^2)\,\delta(x,y,z)\,dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2+y^2)\,\delta(x,y,z)\,dV$.

Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}}\, dxdy$.

We wish to compute $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}} dxdy$. From the bounds we know the region of integration is horizontally simple and has bounds $\{ 0 \le y \le 2,\ y^2 \le x \le 4 \}$. The region can be graphed as:



$x = y^2 \Rightarrow y = \sqrt{x}$

This is the vertically simple region with domain $\{ 0 \le x \le 4,\ 0 \le y \le \sqrt{x} \}$

By Fubini's Theorem, the integral is now: $\int_0^4 \int_0^{\sqrt{x}} e^{x^{\frac{3}{2}}} dydx$, we compute:

$= \int_0^4 e^{x^{\frac{3}{2}}}\, y \Big|_0^{\sqrt{x}} dx = \int_0^4 e^{x^{\frac{3}{2}}} [\sqrt{x} - 0]\, dx$

$= \int_0^4 x^{\frac{1}{2}} e^{x^{\frac{3}{2}}} dx$. Let $u = x^{\frac{3}{2}}$, $du = \frac{3}{2} x^{\frac{1}{2}} dx$

$= \frac{2}{3} \int_{x=0}^{x=4} \frac{3}{2} x^{\frac{1}{2}} e^{x^{\frac{3}{2}}} dx = \frac{2}{3} \int_{x=0}^{x=4} e^{x^{\frac{3}{2}}} \left(\frac{3}{2} x^{\frac{1}{2}} dx \right)$

$= \frac{2}{3} \int_{x=0}^{x=4} e^u du = \frac{2}{3} [e^u]_{x=0}^{x=4}$

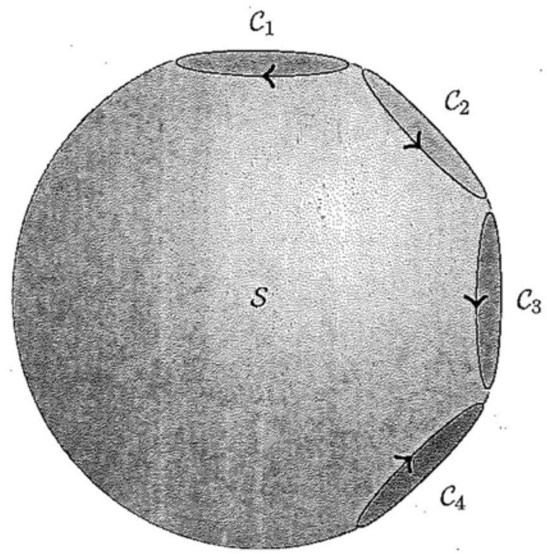
Since $u = x^{\frac{3}{2}}$, $x = 0 \Rightarrow u = 0$ and $x = 4 \Rightarrow u = 4^{\frac{3}{2}} = 8$. Thus, the expression is:

$= \frac{2}{3} [e^u] \Big|_{u=0}^{u=8} = \frac{2}{3}(e^8 - e^0) = \frac{2}{3}(e^8 - 1)$

We have shown that $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}} dxdy$

$= \frac{2}{3}(e^8 - 1)$

2. (8 points) Let $\mathcal{S}$ be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

We wish to compute $\oint_{C_1} F \cdot d\vec{r}$. By Stoke's Theorem: $\iint_S \text{curl} F \cdot d\vec{S} = \oint_{\partial S} F \cdot d\vec{r}$

In the case of this particular sphere:

$$\iint_S \text{curl} \vec{F} \cdot d\vec{S} = \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} - \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r}$$

$\oint_{C_2} \vec{F} \cdot d\vec{r}$ and $\oint_{C_3} \vec{F} \cdot d\vec{r}$ are negative because their orientation is opposite that which we desire. (i.e. walking along C_2 & C_3 in the given orientation would have the "hole" to our left, but we want the surface to our left.)

We rewrite Stokes theorem:

$$\iint_S \text{curl}\,\vec{F}\cdot d\vec{S} = \oint_{C_1}\vec{F}\cdot d\vec{r} - \oint_{C_2}\vec{F}\cdot d\vec{r} - \oint_{C_3}\vec{F}\cdot d\vec{r} + \oint_{C_4}\vec{F}\cdot d\vec{r}$$

We substitute our given values:

$$20 = \oint_{C_1}\vec{F}\cdot d\vec{r} - 305 - 104 + 27$$

We solve for $\oint_{C_1}\vec{F}\cdot d\vec{r}$

$$-7 = \oint_{C_1} F\cdot d\vec{r} - 305 - 104$$

$$-7 = \oint_{C_1} F\cdot d\vec{r} - 409$$

$$402 = \oint_{C_1} F\cdot d\vec{r}$$

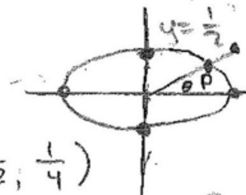
We have proved that $\oint_{C_1}\vec{F}\cdot d\vec{r} = 402$.

$4y^2 = 1 - x^2$
$y = \sqrt{\frac{1-x^2}{4}}$

3. (12 points) Let C be the part of the ellipse $x^2 + 4y^2 = 1$ bewteen $y = 0$ and $y = \frac{1}{2}x$ in the first quadrant. Find $\int_C x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$.

$\frac{1}{4} + 4y^2 = 1$
$4y^2 = \frac{3}{4}$
$y = \frac{3}{16}$ $= \frac{\sqrt{3}}{4}$

The ellipse $x^2 + 4y^2 = 1$ is also $x^2 + \frac{y^2}{(\frac{1}{2})^2} = 1$

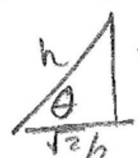


at $x = 0 \Rightarrow 4y^2 = 1 \Rightarrow y = \frac{1}{2}$
at $y = 0 \Rightarrow x^2 = 1 \Rightarrow x = 1$

$P = (\frac{1}{2}, \frac{1}{4})$ The intersection of the ellipse & the line is where the y's are equal.

$x^2 + 4y^2 = 1 \Rightarrow x^2 + 4(\frac{1}{2}x)^2 = 1 \Rightarrow$
$x^2 + \frac{4}{4}x^2 = 1 \Rightarrow 2x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \frac{\sqrt{2}}{2}$
Since $x = \frac{\sqrt{2}}{2}$, $(\frac{\sqrt{2}}{2})^2 + 4(y^2) = 1$
$= \frac{1}{2} + 4y^2 = 1 \Rightarrow 4y^2 = \frac{1}{2} \Rightarrow y^2 = \frac{1}{8} \Rightarrow y = \frac{1}{2\sqrt{2}}$

Thus:



$\frac{1}{2\sqrt{2}} = \frac{\sqrt{2}}{4}$

$h = \sqrt{(\frac{\sqrt{2}}{4})^2 + (\frac{\sqrt{2}}{2})^2} = \sqrt{\frac{1}{8} + \frac{1}{2}} = \sqrt{\frac{5}{8}}$

$\cos = \frac{ADJ}{HYP} = \frac{\sqrt{2}}{2} \cdot \frac{2\sqrt{2}}{\sqrt{5}} = \frac{2}{\sqrt{5}}$

$\sin = \frac{OP}{HYP} = \frac{\sqrt{2}}{4} \cdot \frac{2\sqrt{2}}{5} = \frac{1}{5}$ $\therefore$ $\theta = \arccos(\frac{2}{\sqrt{5}})$ $\theta = \arcsin(\frac{1}{5})$

We parameterize $r(t) = (\cos t, \frac{1}{2}\sin t)$
for $\arccos(\frac{2}{\sqrt{5}}) \le t \le \frac{\pi}{2}$.

Thus by definition: $\int_C f(x,y)\,ds = \int_a^b f(r(t))\,\|r'(t)\|\,dt$

We compute: $x = \cos t$, $y = \frac{1}{2}\sin t$ $x' = \sin t$ $y' = \frac{1}{2}\cos t$

$$\int_{\arcsin(\frac{1}{5})}^{\pi/2} \cos t \sqrt{\frac{1}{4}\cos^2 t + 4(\frac{1}{4}\sin^2 t)} \sqrt{\sin^2 t + \frac{1}{4}\cos^2 t}\, dt$$

$$\int_{\arcsin(\frac{1}{5})}^{\pi/2} \cos t \left(\sin^2 t + \frac{1}{4}\cos^2 t\right) dt$$

$$\int_{\arcsin(1/5)}^{\pi/2} \sin^2 t \cos t + \frac{1}{4}\cos^3 t \, dt$$

$$\int_{\arcsin(1/5)}^{\pi/2} \sin^2 t \cos t + \frac{1}{4}\cos t(1 - \sin^2 t)\, dt$$

$$\int_{\arcsin(1/5)}^{\pi/2} \sin^2 t \cos t + \frac{1}{4}\cos t - \frac{1}{4}\sin^2 t \cos t \, dt$$

$$\int_{\arcsin(1/5)}^{\pi/2} \left(\frac{3}{4}\sin^2 t \cos t + \frac{1}{4}\cos t\right) dt$$

$$= \int_{\arcsin(1/5)}^{\pi/2} \left(\frac{3}{4}\sin^2 t \cos t + \frac{1}{4}\cos t \right) dt$$

let $u = \sin t$ $\quad du = \cos t\, dt$

$t = \arcsin(\frac{1}{5}) \Rightarrow u = \sin(\arcsin(\frac{1}{5})) = \frac{1}{5}$

$t = \pi/2 \Rightarrow u = \sin(\frac{\pi}{2}) = 1$

$$= \int_{1/5}^{1} \frac{3}{4} u^2\, du + \int_{\arcsin(1/5)}^{\pi/2} \cos t\, dt$$

$$= \frac{1}{4}u^3 \Big|_{1/5}^{1} + \sin t \Big|_{\arcsin(1/5)}^{\pi/2}$$

$$= \left[\frac{1}{4}(1) - \frac{1}{4}\left(\frac{1}{5}\right)^3\right] + \left[\sin\frac{\pi}{2} - \sin(\arcsin(\frac{1}{5}))\right]$$

$$= \left[\frac{1}{4} - \frac{1}{4\cdot 125}\right] + \left[1 - \frac{1}{5}\right]$$

$$= \frac{1}{4} - \frac{1}{500} + \frac{4}{5}$$

$$= \frac{500}{2000} - \frac{4}{2000} + \frac{1600}{2000}$$

$$= \frac{1600 + 500 - 4}{2000} = \frac{2100 - 4}{2000} = \frac{2096}{2000}$$

$$= \frac{1048}{1000} = \frac{524}{500} = \frac{262}{250} = \frac{131}{125}$$

We have shown that $\int_C x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$

$= \frac{131}{125}$ for the calculated curve C.

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2 + y^2 + z^2 \leq \frac{1}{100}$ and $\sqrt{3}z \leq -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x, y, z) = 5\,\mathrm{kg\,m^{-3}}$.

   (a) Write $\mathcal{W}$ using spherical coordinates.

   (b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*

a) We square both sides of $\sqrt{3}z \leq -\sqrt{x^2+y^2}$

$\Rightarrow 3z^2 \leq x^2+y^2$ which is a cone.

The cone and the sphere intersect at the same z-coordinate, so:

$x^2+y^2+z^2 \leq \frac{1}{100} \Rightarrow 3z^2+z^2 \leq \frac{1}{100} \Rightarrow 4z^2 \leq \frac{1}{100}$

$\Rightarrow z^2 \leq \frac{1}{400} \Rightarrow z = \frac{1}{20}$ which is a circle at the height of $\frac{1}{20}$. But since $\sqrt{3}z \leq -\sqrt{x^2+y^2}$, it is the cone in the lower hemisphere, that is z is actually $-\frac{1}{20}$. At $z = -\frac{1}{\sqrt{20}}$

$x^2+y^2+\left(-\frac{1}{\sqrt{20}}\right)^2 \leq \frac{1}{100}$

$x^2+y^2+\frac{1}{20} \leq \frac{1}{100} \Rightarrow x^2+y^2+\frac{5}{100} \leq \frac{1}{100}$

$\mathcal{W}: \{ 0 \leq \theta \leq 2\pi,\ 0 \leq \phi \leq \frac{\pi}{3},$

b) The moment of inertia around the z-axis is defined as $I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dV$.
Since $\delta(x,y,z) = 5\ kg/m^3$, $I_z = 5\iiint_W \delta(x,y,z)\,dV$,
The units of the answer are $kg \cdot m^2$.
We change to spherical coordinates so: $\frac{1}{100} = r^2 \Rightarrow r = \frac{1}{10}$
$dV = r^2 \sin\phi \, dr\, d\theta\, d\phi$.

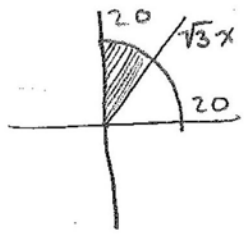
Assume W is a sphere w/ radius $\frac{1}{\sqrt{100}}$.
Then $W: \{ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi,\ 0 \le r \le \frac{1}{10} \}$
let $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = \cos\phi$

$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1/10} (r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi)\,5r^2\sin\phi\,dr\,d\theta\,d\phi$$
$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1/10} (r^2\sin^2\phi)\,5r^2\sin\phi\,dr\,d\theta\,d\phi$$
$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1/10} r^4\sin^3\phi\,dr\,d\theta\,d\phi$$
$$I_z = \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^{1/10} r^4\,dr\right)$$
$$I_z = (2\pi)\left(\int_0^{\pi}\sin\phi(1-\cos^2\phi)\,d\phi\right)\left(\frac{1}{5}r^5\Big|_0^{1/10}\right)$$
$$I_z = (2\pi)\left(\frac{1}{5}\left(\frac{1}{10}\right)^5\right)\left(\int_0^{\pi}\sin\phi - \cos^2\phi\sin\phi\,d\phi\right)$$

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \sqrt{3}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

$$p_{X,Y}(x,y) = \begin{cases} \frac{3}{25}\frac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.



We wish to find $\mathbb{E}[\sqrt{x^2+y^2}]$.

It is given that:

$$E[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, p_{xy}(x,y)\, dx\, dy.$$

Since $p_{xy}(x,y) = 0$ for $(x,y) \notin D$, we can change the bounds of integration from $x \in [-\infty, \infty]$ and $y \in [-\infty, \infty]$ to the bounds of D. We compute D in polar coordinates. (also $dx\,dy = r\,dr\,d\theta$)

$\sqrt{3}x$ and $x^2+y^2 = 400$ intersect at:

$$x^2 + (\sqrt{3}x)^2 = 400 \Rightarrow x^2 + 3x^2 = 400$$

$$4x^2 = 400 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10 \Rightarrow x = +10.$$

Thus: $\cos\theta = \frac{10}{20} = \frac{1}{2}$ $\therefore$ $\theta = \frac{\pi}{3}$.

This means D can be written as:

$D: \{ 0 \le r \le 20, \frac{\pi}{3} \le \theta \le \frac{\pi}{2} \}$. Thus:

$$\mathbb{E}[\sqrt{x^2+y^2}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2}\, \frac{3}{25}\frac{x^2 y}{(x^2+y^2)^{3/2}}\, dx\, dy$$

$x = r\cos\theta$, $y = r\sin\theta$ $\therefore$ $r = \sqrt{x^2+y^2}$

$$\mathbb{E}[\sqrt{x^2+y^2}] = \iint_D r\, \frac{3}{25}\, \frac{r^2\cos\theta\, r\sin\theta}{r^3}\, r\,dr\,d\theta$$

$$= \iint_D \frac{3}{25}\, \frac{r^5\cos^2\theta\sin\theta}{r^3}\, dr\,d\theta$$

$$= \iint_D \frac{3}{25}\, r^2\cos^2\theta\sin\theta\, dr\,d\theta$$

$$= \int_0^{20}\int_{\pi/3}^{\pi/2} \frac{3}{25} r^2 \cos^2\theta \sin\theta \, d\theta \, dr$$

$$= \frac{3}{25}\left(\int_0^{20} r^2 dr\right)\left(\int_{\pi/3}^{\pi/2} \sin\theta \cos^2\theta \, d\theta\right)$$

let $u = \cos\theta$
$du = -\sin\theta d\theta$
$\theta = \pi/3 \Rightarrow u = \frac{1}{2}$
$\theta = \pi/2 \Rightarrow u = 0$

$$= \frac{3}{25}\left(\frac{1}{3} r^3 \Big|_0^{20}\right)\left(-\int_{1/2}^{0} \cos^2\theta(-\sin\theta d\theta)\right)$$

$$= \frac{\cancel{3}}{25}\left(\frac{1}{\cancel{3}}[20]^3\right)\left(+\int_0^{1/2} u^2 du\right)$$

$$= \frac{20^3}{25}\left(\frac{1}{3}u^3 \Big|_0^{1/2}\right)$$

$$= \frac{8000}{25}\left(\frac{1}{3}\left(\frac{1}{2}\right)^3 - \cancel{\frac{1}{3}(0)^3}\right)$$

$$= \frac{8000}{25}\left(\frac{1}{24}\right)$$

$$= \frac{800\cancel{0}}{60\cancel{0}} = \frac{80}{6} = \frac{40}{3}$$

$400 \times 20 = 8000$

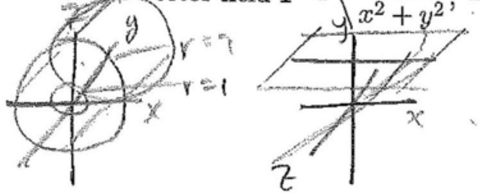
$25 \overline{)8000} = 320$

$\left(\frac{1}{2}\right)^3 = \frac{1}{8}$

$25 \times 24 = 600$

The expected value $E\left[\sqrt{x^2+y^2}\right]$ for the given probability density function is $\frac{40}{3}$.

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $x^2+z^2=1$, $x^2+z^2=9$ and the planes $y=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F}=\left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 3z\right\rangle$ across $\mathcal{S}$.



We wish to compute $\iint_S \vec{F}\cdot d\vec{S}$ for the given vector field F across S. By the divergence theorem: $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div}\vec{F}\, dV$.

We compute $\operatorname{div} F = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial z}(3z)$

$= \frac{\partial}{\partial x}\left[(x^2+y^2)^{-1}y\right] + \frac{\partial}{\partial y}\left[(x^2+y^2)^{-1}(-x)\right] + 3$

$= -2xy(x^2+y^2)^{-2} + 2xy(x^2+y^2)^{-2} + 3$

$= \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 3 = 3$. Thus: $\operatorname{div}\vec{F} = 3$.

We parameterize W in cylindrical coordinates:

$W: \{0 \le \theta \le 2\pi,\ 1 \le r \le 3,\ r\cos\theta \le y \le 3\}$

where $x = r\cos\theta$, $z = r\sin\theta$, $y = y$.

Since $\iint_S \vec{F}\cdot d\vec{S} = \iiint_W \operatorname{div} F\, dV$, we compute:

$\int_0^{2\pi}\int_1^3\int_{r\cos\theta}^3 3\, dy\, dr\, d\theta = \int_0^{2\pi}\int_1^3 3y\Big|_{r\cos\theta}^{y=3} dr\, d\theta$

$= \int_0^{2\pi}\int_1^3 3\left[3 - r\cos\theta\right] dr\, d\theta = \int_0^{2\pi}\int_1^3 9 - 3r\cos\theta\, dr\, d\theta$

$= \int_0^{2\pi} 9r - \frac{3}{2}r^2\cos\theta \Big|_1^3 d\theta$

$= \int_0^{2\pi}\left(9(3) - \frac{3}{2}(3)^2\cos\theta - 9(1) + \frac{3}{2}(1)^2\cos\theta\right) d\theta$

$= \int_0^{2\pi}\left(27 - \frac{27}{2}\cos\theta - 9 + \frac{3}{2}\cos\theta\right) d\theta$

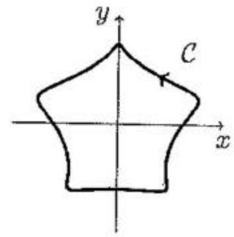
$= \int_0^{2\pi}\left(18 - 12\cos\theta\right) d\theta$

$= \left[18\theta - 12\sin\theta\right]_0^{2\pi}$

$= 18(2\pi) - \cancel{12\sin(2\pi)} - \cancel{18(0)} + \cancel{12\sin 0} = 36\pi$

We have thus shown that the flux through the given surface S is equal to 36π.

7. (12 points) Let $\mathcal{C}$ be the curve



Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \cos(x^3) + ye^{xy},\ \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle.$$

(*Hint: Try writing* $\mathbf{F}$ *as a sum of two vector fields that we know how to integrate around* $\mathcal{C}$.)

$\vec{F}(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle + \left\langle \cos(x^3) + ye^{xy}, e^{e^y} + xe^{xy} \right\rangle$

By the fundamental theorem of vector line integrals: $\oint_C F \cdot dr = 0$ if F has a conservative function f such that $F = \nabla f$

Thus: $\oint_C F \cdot dr = \oint_C (\vec{F}_A + \vec{F}_B) \cdot d\vec{r}$

where $F_A = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$, $F_B = \left\langle \cos(x^3) + ye^{xy}, e^{e^y} + xe^{xy} \right\rangle$

We compute $\operatorname{curl}_z F_B = \left(\frac{\partial F_{B2}}{\partial x} - \frac{\partial F_{B1}}{\partial y} \right)$

$= \left[\frac{\partial}{\partial x}\left(\cancel{e^{e^y}} + xe^{xy}\right) - \frac{\partial}{\partial y}\left(\cancel{\cos(x^3)} + ye^{xy}\right) \right]$

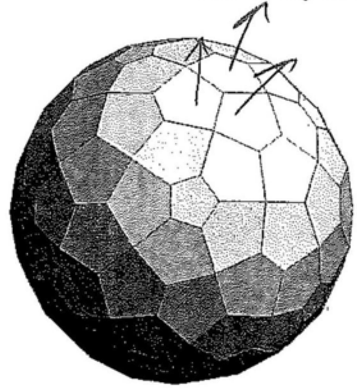
$= \left[(e^{xy} + xye^{xy}) - (e^{xy} + xye^{xy}) \right]$

$= 0$

Since $\operatorname{curl}_z F = 0$ and $\vec{F}$ is defined for all $(x,y) \in \mathbb{R}^2$, there exists a potential function f such that $\nabla f = F$. Also, $\oint_C \vec{F} \cdot d\vec{r} = 0$ for a conservative F (i.e. where $\nabla f = F$). Thus: We neglect F_B in $\oint (F_A + F_B) \cdot d\vec{r}$.

The integral $\oint_C F \cdot dr$ is now $\oint_C \vec{F_A} \cdot d\vec{r}$ where F_A is the vortex field. The property of the vortex field is that $\oint_C \vec{F}_{vortex} \cdot d\vec{r} = 2\pi n$ where n is the number of counterclockwise loops around the origin. Since, by the picture, C makes one loop around the origin in the ccw direction, $\oint_C \vec{F_A} \cdot d\vec{r} = 2\pi$, and since $\oint_C \vec{F} \cdot d\vec{r} = \oint_C F_A \cdot dr$, we have shown that $\oint_C \vec{F} \cdot d\vec{r} = 2\pi$ for the given closed loop C.

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{a}_j$ so that $\mathbf{a}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{a}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

a) We wish to show that $\|F\|^2 = \iint_S \vec{F} \cdot d\vec{s}$.
Since $\|F\|^2 = F \cdot F$, $F \cdot F = \iint_S \vec{F} \cdot d\vec{s}$
By definition: $\iint_S F \cdot d\vec{s} = \iint_{S_1} \vec{F} \cdot d\vec{s} + \cdots + \int_{S_k} \vec{F} \cdot d\vec{s}$
Since $N_1, N_2, \ldots, N_j$ are constant on S, F is constant on S. By the divergence theorem: $\iint_S F \cdot ds = \iiint_W \operatorname{div} F \, dV$.
The divergence of a constant vector field F yields a constant scalar field which we denote $f = \operatorname{div} F$.
Since f is constant:
$\iiint_W \operatorname{div} F \, dV = f \iiint_W 1 \, dV = f \cdot \text{Volume}(W)$

b) Assuming that $\|F\|^2 = \iint_S \vec{F} \cdot d\vec{S}$, we wish to show that $\vec{F} = \vec{0}$. It is true that:

$\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \dots + \iint_{S_k} \vec{F} \cdot d\vec{S}$

Since $N_1, N_2, \dots N_K$ are constant on S, F is constant on S. By the divergence theorem: $\iint_S \vec{F} \cdot d\vec{S} = \iiint_{\mathcal{W}} \operatorname{div} F = 0$.

However, the divergence of a constant vector field yields 0 so $\operatorname{div} F = 0$.

Thus: $\iiint_{\mathcal{W}} 0 = 0 = \iint_S \vec{F} \cdot d\vec{S}$

Since $\iint_S F \cdot ds = \|F\|^2$,

$0 = \|F\|^2$

and thus F must be 0.