Math 32B Final Counterclockwise

[redacted]

TOTAL POINTS

75 / 90

QUESTION 1

1 Fubini's Theorem 6 / 6

✓ **+ 6 pts Correct answer (2/3)(e^8 - 1).**

+ **2 pts** (Partial credit) New x limits are 0 to 4.
+ **2 pts** (Partial credit) New y limits are 0 to sqrt(x).
+ **1 pts** (Partial credit) New y integral is sqrt(x)*exp(x^(3/2)).
+ **1 pts** (Partial credit, only applies if new limits are incorrect) Reasonably correct picture.
+ **0 pts** No points.
+ **3 pts** (Partial credit) Incorrect limits: 0 <= x <= 4, sqrt(x) <= y <= 2

QUESTION 2

2 Stokes' Theorem 8 / 8

✓ **+ 8 pts Correct answer 402.**

+ **4 pts** (Partial credit) Answer for _inward_ pointing normal 362.
+ **0 pts** No points.
+ **7 pts** (Partial credit) Correct method and orientations, but arithmetic error
+ **3 pts** (Partial credit) Line integral over C1 is equal to sum of line integrals and surface integral, with some (incorrect) choice of signs.
+ **2 pts** (Partial credit, only if no other points apply) Mention or state Stoke's theorem.

QUESTION 3

3 Line integral 10 / 12

✓ **+ 4 pts Correct parametrization**

+ **2 pts** Partial credits for parametrization

✓ **+ 4 pts Correct integral formula**

+ **2 pts** Partial credits for integral
+ **4 pts** Correct calculation

✓ **+ 2 pts Partial credits for calculation**

+ **1 pts** Almost makes no sense.
+ **0 pts** Nothing correct

- **1 pts** Tiny calculation error

QUESTION 4

4 Moment of inertia 11 / 14

✓ **+ 1 pts a) Correct limits $$0\leq\rho\leq\frac1{10}$$**

✓ **+ 1 pts a) Correct limits $$0\leq\theta<2\pi$$**

+ **1 pts** a) Correct upper bound $$\phi\leq \pi$$
+ **2 pts** a) Correct lower bound $$\phi\geq \frac{2\pi}3$$

✓ **+ 1 pts b) Correctly using part (a) to obtain limits (credit given even if limits wrong, provided they are consistent)**

✓ **+ 1 pts b) Correct integrand $$5(x^2+y^2)$$ (must substitute $$\delta=5$$ into formula from formula sheet to gain credit)**

✓ **+ 2 pts b) Correctly converting $$x^2 + y^2$$ to $$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$ in spherical coordinates**

✓ **+ 1 pts b) Correctly simplifying $$5(x^2+y^2)$$ to $$5\rho^2\sin^2\phi$$**

✓ **+ 2 pts b) Correct Jacobian $$\rho^2\sin\phi$$ in spherical coordinates**

+ **1 pts** b) Correct answer of $$\frac{\pi}{240000}\,\mathrm{kg}\,\mathrm{m}^2$$ (units required for points, only awarded if rest of computation correct)

✓ **+ 1 pts Solution thoroughly explained, using full sentences**

✓ **+ 1 pts Correct picture(s) of region (bonus point, only awarded if points lost elsewhere)**

+ **0 pts** No credit due

QUESTION 5

5 Probability 14 / 14

✓ + **14 pts** **Full points**

+ **0 pts** No points
+ **2 pts** Correctly labeled region (all or nothing)
+ **3 pts** Correctly set-up integral (max 6 pts)
+ **2 pts** Correctly set-up integral
+ **1 pts** Correctly set-up integral
+ **3 pts** Evaluation of integral (max 6 pts)
+ **2 pts** Evaluation of integral
+ **1 pts** Evaluation of integral

QUESTION 6

6 Divergence Theorem 14 / 14

✓ + **4 pts** **Correct divergence**

✓ + **7 pts** **Correct parametrization of $$ \mathcal{W}$$**

✓ + **3 pts** **Correct evaluation of correct triple integral (implicit in the grading process was that this rubric item meant that you could have also correctly computed the volume using high school geometry)**

+ **2 pts** Bonus: Drew accurate picture (must include both cylinders and both planes, and accurate portrayal of their intersections [the larger cylinder and two planes meet in a single point])
+ **0 pts** No credit

QUESTION 7

7 Vector line integral 5 / 12

+ **4 pts** Write F as a sum of vortex field and a conservative field
+ **2 pts** Vortex field has integral 2pi over this C

✓ + **2 pts** **Compute curl_z F_2 or show F_2 is conservative**

✓ + **3 pts** **Conclude (e.g. by Green's theorem or using that F_2 is conservative) that the integral over C of F_2 is 0**

+ **1 pts** Arrive at correct answer, 2pi, by valid method
+ **0 pts** Incorrect
+ **2 pts** Mostly correct argument that integral of F_2 is 0
+ **1 pts** curl_z F_2 minor error

(1) can't apply Green's theorem to F_1 because of singularity; (2) "More matter with less art."

QUESTION 8

8 Surface integral 7 / 10

✓ + **3 pts** **Decompose flux integral**

+ **1 pts** Partial credit for decomposition

✓ + **2 pts** **Do component integrals**

+ **1 pts** Partial credit for component integrals

✓ + **1 pts** **Combine integrals**

+ **2 pts** Used divergence theorem (part (b))
+ **1 pts** Correct (and justified) div(F) (part (b))

✓ + **1 pts** **Clear and well-explained solution**

+ **0 pts** No credit due

Too many words—you don't need to give that much detail. Better too much than too little, though. You need to show that the normals will cancel; that's the point of the problem—it's not obvious.

Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: [redacted]
SID: [redacted]
TA Section: [redacted]

Time Limit: 180 Minutes

Version 5

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M} \right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x, y, z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x, y, z)\, dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}, z_{\text{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M} \right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2 + y^2)\, \delta(x, y, z)\, dV$.

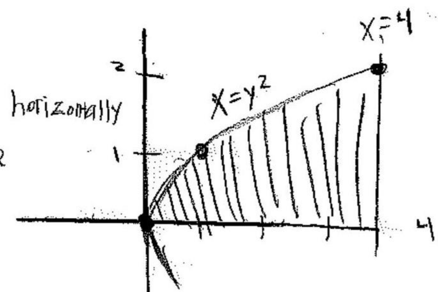
Probability formulas

- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \leq b$ is $\mathbb{P}[a < X \leq b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}}\,dx\,dy$.

We compute this double integral by applying Fubini's theorem, to change the order of integration, so that the integral is easier to compute.

Sketching the domain of the region:



We can see that the region is bounded horizontally by $x=4$ on the right and $x=y^2$ on the left. We now want to find a domain that is vertically simple to be able to integrate over. $y=\pm\sqrt{x}$, but only want positive because $y \geq 0$. As well, let x vary from 0 to 4 to get $D=\{0 \leq x \leq 4, 0 \leq y \leq \sqrt{x}\}$

$0 \leq y \leq 2$

Now, applying Fubini's Theorem, we can integrate over the region.

We compute $\int_0^4 \int_0^{\sqrt{x}} e^{x^{3/2}}\,dy\,dx = \int_0^4 y e^{x^{3/2}} \Big]_0^{\sqrt{x}} dx = \int_0^4 x^{1/2} e^{x^{3/2}}\,dx$, now we let $u = x^{3/2}$ and $du = \frac{3}{2}x^{1/2}dx$.

$= \frac{2}{3}\int_{u=0}^{u=8} e^u\,du = \frac{2}{3}e^u \Big]_{u=0}^{u=8} = \frac{2}{3}e^8 - \frac{2}{3}e^0 = \boxed{\frac{2}{3}(e^8-1)}$

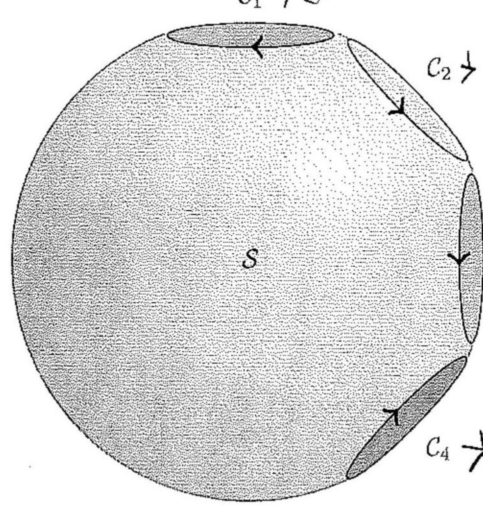
$u(0)=0$

$u(4) = 4^{3/2} = 2^3 = 8$

Therefore, by applying Fubini's Theorem, we were able to compute the value of the double integral by changing the order of integration.

2. (8 points) Let S be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves C_1, C_2, C_3, C_4 oriented as in the following picture:

$C_1 \to$ correctly oriented



$C_2 \to$ incorrectly oriented

$C_3 \to$ incorrectly oriented.

$C_4 \to$ correctly oriented

Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_S \operatorname{curl}\mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{C_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{C_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r}$.

It looks like we should apply Stokes' Theorem since $\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$ by the theorem.

We now need to determine if the curves are properly oriented, to solve for $\oint_{C_1} \vec{F} \cdot d\vec{r}$. C_1 is oriented correctly, while C_2 is oriented backwards, C_3 is oriented backwards and C_4 is oriented correctly for S to have outward normals. Therefore we need to subtract the curves that are incorrectly oriented to be able to apply Stokes' Theorem.

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = +\oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r} - \oint_{C_3} \vec{F} \cdot d\vec{r} + \oint_{C_4} \vec{F} \cdot d\vec{r} \text{ by Stokes' Theorem}$$

$$20 = \oint_{C_1} \vec{F} \cdot d\vec{r} - 305 - 104 + 27$$

$$\begin{array}{r} 305 \\ 104 \\ \hline 409 \\ 27 \\ 382 \end{array}$$

$$20 = \oint_{C_1} \vec{F} \cdot d\vec{r} - 382$$

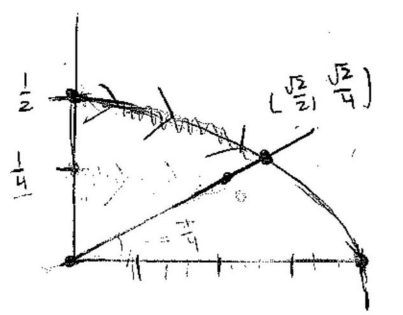
$$\boxed{\oint_{C_1} \vec{F} \cdot d\vec{r} = 402}$$

Therefore, by applying Stokes' Theorem and looking at the orientation of the boundary curves, we were able to calculate $\oint_{C_1} \vec{F} \cdot d\vec{r}$.

3. (12 points) Let C be the part of the ellipse $x^2 + 4y^2 = 1$ bewteen $y = 0$ and $y = \frac{1}{2}x$ in the first quadrant. Find $\int_C x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$.

$ds = \|\vec{r}'(t)\| \rightarrow$ not $\hat{n}$ because it is just ds and not $d\vec{s}$

First sketch the region we want to integrate over.



$Y = \frac{1}{2}x$

$X^2 + 4(\frac{1}{2}X)^2 = 1$

$X^2 + X^2 = 1$

$2X^2 = 1$

$X^2 = \frac{1}{2}$

$X = \sqrt{\frac{1}{2}}$, $Y = \frac{1}{2}\sqrt{\frac{1}{2}} = \sqrt{\frac{1}{8}} = \frac{1}{\sqrt{8}} = \frac{\sqrt{8}}{8} = \frac{2\sqrt{2}}{8} = \frac{\sqrt{2}}{4}$

We should try to parametrize the curve, so that we can compute the integral..

Normally, $\vec{r}(t) = \langle r\cos t, r\sin t\rangle$, r varies here in this problem... and the orientation is opposite, so $\vec{t}$ will need to change the order to $\vec{r}(t) = \langle r\sin t, r\cos t\rangle$

When $t = 0$, we should get the coordinates $\langle 0, \frac{1}{2}\rangle$, so $\vec{r}(t) = \langle r\sin t, \frac{1}{2}\cos t\rangle$ and when $t = \frac{\pi}{2}$ we should get $\langle 1, 0\rangle$, which then gives us our parametrization for the curve! However our curve does not go from $0 \le t \le \pi/2$ it goes to $0 \le t \le ?$

$\vec{r}(t) = \langle \sin t, \frac{1}{2}\cos t\rangle$ for $0 \le t \le ?$ Now, we need to find the upper limit for t such that $\vec{r}(?) = \langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{4}\rangle$, so, we let $\sin t = \frac{\sqrt{2}}{2}$, so $t = \frac{\pi}{4}$ ✓

and we let $\frac{1}{2}\cos t = \frac{\sqrt{2}}{4}$, so $\cos t = \frac{\sqrt{2}}{2}$, $t = \frac{\pi}{4}$ ✓

Therefore, our complete parametrization of the curve is $\vec{r}(t) = \langle \sin t, \frac{1}{2}\cos t\rangle$ for $0 \le t \le \frac{\pi}{4}$.

Now, we compute $\vec{r}'(t) = \langle \cos t, -\frac{1}{2}\sin t\rangle$, and $ds = \|\vec{r}'(t)\|dt = \|\langle\cos t, -\frac{1}{2}\sin t\rangle\| = \sqrt{\cos^2 t + \frac{1}{4}\sin^2 t}\, dt$

We also find our $f(\vec{r}(t)) = \sin t\sqrt{\frac{1}{4}\sin^2 t + 4(\frac{1}{4})\cos^2 t} = \sin t\sqrt{\frac{1}{4}\sin^2 t + \cos^2 t}$

Therefore, $\int_C x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds = \int_0^{\pi/4} \sin t\sqrt{\frac{1}{4}\sin^2 t + \cos^2 t}\sqrt{\cos^2 t + \frac{1}{4}\sin^2 t}\, dt = \int_0^{\pi/4} \sin t(\frac{1}{4}\sin^2 t + \cos^2 t)\, dt$

$= \int_0^{\pi/4} \frac{1}{4}\sin^3 t + \sin t\cos^2 t\, dt$, lets compute these integrals separately.

From the previous page, $\int_C x\sqrt{\frac{1}{4}x^2+4y^2}\,ds = \int_0^{\pi/4} \frac{1}{4}\sin^3 t + \sin t\cos^2 t\,dt$

and if we compute the integral of the two terms separately:

$\frac{1}{4}\int_0^{\pi/4}\sin^3 t\,dt = \frac{1}{4}\int_0^{\pi/4}\sin t(1-\cos^2 t)\,dt = \frac{1}{4}\int_0^{\pi/4}\sin t\,dt - \frac{1}{4}\int_0^{\pi/4}\sin t\cos^2 t\,dt$, compute the integrals separately once again,

$\frac{1}{4}\int_0^{\pi/4}\sin t\,dt = -\frac{1}{4}\cos t\Big]_0^{\pi/4} = -\frac{1}{4}\left(\frac{\sqrt{2}}{2}-1\right) = \frac{1}{4}-\frac{\sqrt{2}}{8}$

We compute $-\frac{1}{4}\int_0^{\pi/4}\sin t\cos^2 t\,dt$, let $u=\cos t$, $du=-\sin t$, $=\frac{1}{4}\int_0^{\pi/4}u^2\,du = \frac{1}{12}u^3\Big]_0^{\pi/4} = \frac{1}{12}\cos^3 t\Big]_0^{\pi/4} = \frac{1}{12}\left(\frac{2\sqrt{2}}{8}-1\right)$

So $\frac{1}{4}\int_0^{\pi/4}\sin^3 t\,dt = \frac{1}{4}-\frac{\sqrt{2}}{8}+\frac{1}{12}\left(\frac{\sqrt{2}}{4}-1\right)$

Now we compute $\int_0^{\pi/4}\sin t\cos^2 t\,dt =$ let $u=\cos t$, $du=-\sin t\,dt$ $-\int_0^{\pi/4}u^2\,du = -\frac{1}{3}u^3\Big]_0^{\pi/4} = -\frac{1}{3}(\cos^3 t)\Big]_0^{\pi/4} = -\frac{1}{3}\left(\frac{2\sqrt{2}}{8}-1\right)$

So our total integral is $\int_0^{\pi/4}\frac{1}{4}\sin^3 t+\sin t\cos^2 t\,dt = \frac{1}{4}-\frac{\sqrt{2}}{8}+\frac{1}{12}\left(\frac{\sqrt{2}}{4}-1\right)-\frac{1}{3}\left(\frac{\sqrt{2}}{4}-1\right)$

$= \frac{1}{4}-\frac{\sqrt{2}}{8}+\frac{\sqrt{2}}{48}-\frac{1}{12}-\frac{\sqrt{2}}{12}+\frac{1}{3} = \frac{3}{12}+\frac{4}{12}-\frac{1}{12}+\frac{\sqrt{2}}{48}-\frac{4\sqrt{2}}{48}-\frac{6\sqrt{2}}{48} = \boxed{\frac{1}{2}-\frac{3\sqrt{2}}{16}}$

Therefore, by parametrizing the curve of the part of the ellipse in the correct direction, we were able to compute the line integral.

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2+y^2+z^2 \le \frac{1}{100}$ and $\sqrt{3}z \le -\sqrt{x^2+y^2}$, where distance is measured in meters, and has constant density $\delta(x,y,z) = 5\,\mathrm{kg\,m^{-3}}$.

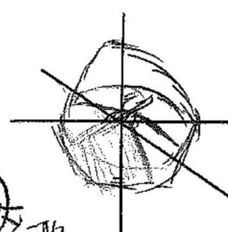
neg pos

(a) Write $\mathcal{W}$ using spherical coordinates.

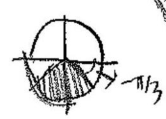
(b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*

a)

Z need s to be negative!



In 2D



We are looking at circles in the sphere, so $0 \le \theta \le 2\pi$!

Therefore, our region can be written as

$$\boxed{W = \{\ 0 \le \theta \le 2\pi,\ \frac{\pi}{2} \le \phi \le \frac{2\pi}{3},\ 0 \le \rho \le \frac{1}{10}\ \}}$$

In spherical coordinates: $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$

so,

$$\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi + \rho^2\cos^2\phi \le \frac{1}{100}$$

$$\rho^2\sin^2\phi + \rho^2\cos^2\phi \le \frac{1}{100}$$

$\rho^2 \le \frac{1}{100}$, so ρ varies between 0 and $\frac{1}{10}$.

must be negative must be positive

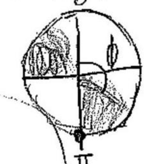
$$\sqrt{3}\rho\cos\phi \le -\sqrt{\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi}$$

$$\sqrt{3}\rho\cos\phi \le -\sqrt{\rho^2\sin^2\phi}$$

$$\sqrt{3}\rho\cos\phi \le -\rho\sin\phi$$

$$\sqrt{3}\cos\phi \le -\sin\phi$$

$\sqrt{3} \le -\tan\phi$ $\rightarrow$ $\tan\phi$ must be negative!



$\tan\phi = \sqrt{3}$

$\phi = -\frac{\pi}{3}$

$$\frac{\pi}{2} \le \phi \le \frac{2\pi}{3}$$

because $0 \le \phi \le \pi$, namely, so above is the additional constraint

b) The moment of inertia about the Z-axis is

$I_z = \iiint_W (x^2+y^2)\,\delta(x,y,z)\,dV$, so $x^2+y^2 = \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi = \rho^2\sin^2\phi$

$5\,\mathrm{kg/m^3}$ and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, so

$$I_z = \int_0^{2\pi}\int_{\pi/2}^{2\pi/3}\int_0^{\frac{1}{10}} 5\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta = 5\int_0^{2\pi} d\theta \int_{+\frac{\pi}{2}}^{+\pi/3}\int_0^{\frac{1}{10}} \rho^4\sin^3\phi\,d\rho\,d\phi = 10\pi\int_{+\frac{\pi}{2}}^{+\pi/3} \frac{1}{5}\rho^5\sin^3\phi\Big]_0^{1/10} d\phi$$

$$= 2\pi(10^{-5})\int_{+\frac{\pi}{2}}^{+\pi/3}\sin^3\phi\,d\phi = 2\pi(10^{-5})\int_{+\frac{\pi}{2}}^{+\pi/3}\sin\phi(1-\cos^2\phi)\,d\phi = 2\pi(10^{-5})\int_{+\frac{\pi}{2}}^{+\pi/3}\sin\phi - \int_{+\frac{\pi}{2}}^{+\pi/3}\sin\phi\cos^2\phi\,d\phi$$

let $u = \cos\phi$, $du = -\sin\phi\,d\phi$

↗ Work for this integral was done in the previous problem twice!

$$= 2\pi(10^{-5})\left(-\cos\phi + \frac{1}{3}(\cos^3\phi)\Big]_{-\pi/2}^{-\pi/3}\right) = 2\pi(10^{-5})\left(-\left(-\frac{1}{2}\right) + \frac{1}{3}\left(-\frac{1}{2}\right)^3 - 0\right) = 2\pi(10^{-5})\left(\frac{1}{2} - \frac{1}{3}\left(\frac{1}{8}\right)\right)$$

$$2\pi(10^{-5})\left(\frac{1}{2} - \frac{1}{24}\right) = 2\pi(10^{-5})\left(\frac{12}{24} - \frac{1}{24}\right) = \boxed{\frac{11\pi}{12}(10^{-5})\ \mathrm{kg.}}$$

This answer seems very small, but the region is also very small, so it makes sense.

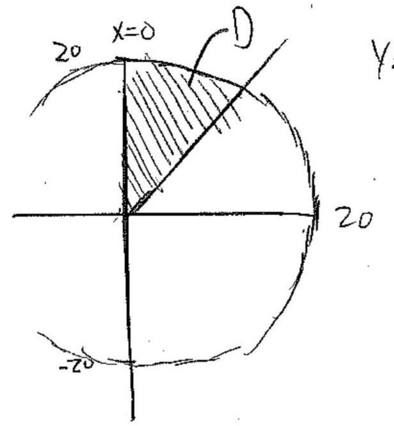
Therefore we were able to compute the moment of inertia of the region given in the problem W about the Z-axis.

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \sqrt{3}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

$$p_{X,Y}(x,y) = \begin{cases} \dfrac{3}{25}\dfrac{x^2 y}{(x^2+y^2)^{\frac{3}{2}}} & \text{if } (x,y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2+Y^2}$. Find $\mathbb{E}[\sqrt{X^2+Y^2}]$.

Domain sketch:



$Y = \sqrt{3}X$

distance $= \sqrt{x^2+y^2}$

We know. $\mathbb{E}[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, P_{X,Y}(x,y)\,dx\,dy$

It seems that changing to polar coordinates will be much easier because everything appears to have some form of radial geometry.

$X = r\cos\theta$, $Y = r\sin\theta$, $r = \sqrt{x^2+y^2}$, so our probability density function becomes

$P_{X,Y}(X,Y) = \begin{cases} \frac{3}{25}\frac{r^3 \sin\theta\cos^2\theta}{r^3} & \text{if } (r,\theta) \in D \\ 0 & \text{otherwise} \end{cases}$ $\quad P_{R,\theta}(r,\theta) = \begin{cases} \frac{3}{25}\sin\theta\cos^2\theta & \text{if } (r,\theta) \in D \\ 0 & \text{otherwise.} \end{cases}$

Our function $f(x,y) = \sqrt{x^2+y^2}$, becomes $f(r,\theta) = r$ and $dx\,dy = r\,dr\,d\theta$. Now we need to find our domain to integrate over.

$x^2+y^2 = 400$

$r^2 \le 400$, so : $0 \le r \le 20$, $\quad x = 0$ $\quad r\cos\theta = 0$, so $\cos\theta$ has a bound at $\frac{\pi}{2}$, as well

$Y = \sqrt{3}x$, so $r\sin\theta = \sqrt{3}r\cos\theta$

$\tan\theta = \sqrt{3}$, so $\theta = \frac{\pi}{3}$, Therefore $\frac{\pi}{3} \le \theta \le \frac{\pi}{2}$ so $D = \{0 \le r \le 20, \frac{\pi}{3} \le \theta \le \frac{\pi}{2}\}$

Now, we are able to actually write out our integral for the expected value as the sum of integrals,

$\mathbb{E}[f(x,y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,P_{XY}(x,y)\,dx\,dy = \underbrace{\int_{-\frac{3\pi}{2}}^{\pi/3}\int_{0}^{\infty} f(r,\theta)\,P_{r,\theta}(r,\theta)\,r\,dr\,d\theta}_{0} + \underbrace{\int_{\frac{\pi}{3}}^{\pi/2}\int_{20}^{\infty} f(r,\theta)\,P_{r,\theta}(r,\theta)\,r\,dr\,d\theta}_{0} + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}}\int_{0}^{20} f(r,\theta)\,P_{r,\theta}(r,\theta)\,r\,dr\,d\theta$

because $P_{(r,\theta)}(r,\theta) = 0$ when $(r,\theta) \notin D$

Therefore $\mathbb{E}[r] = \int_{\frac{\pi}{3}}^{\pi/2}\int_0^{20} r\left(\frac{3}{25}\sin\theta\cos^2\theta\right) r\,dr\,d\theta$

$$= \frac{3}{25}\int_{\frac{\pi}{3}}^{\pi/2}\int_0^{20} r^2\sin\theta\cos^2\theta\,dr\,d\theta = \frac{1}{25}\int_{\frac{\pi}{3}}^{\pi/2} r^3\sin\theta\cos^2\theta\Big]_0^{20} d\theta = \frac{20^3}{25}\int_{\frac{\pi}{3}}^{\pi/2}\sin\theta\cos^2\theta\,d\theta$$

let $u=\cos\theta$, $du=-\sin\theta\,d\theta$, so $= \frac{-20^3}{25}\int_{\frac{\pi}{3}}^{\pi/2} u^2\,du = \frac{-20^3}{75}\,u^3\Big]_{\pi/3}^{\pi/2} = \frac{-20^3}{75}\cos^3\theta\Big]_{\pi/3}^{\pi/2}$

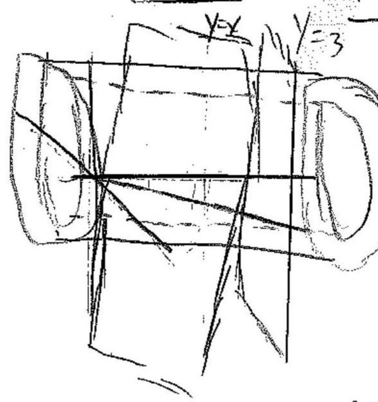
$$= \frac{-20^3}{75}\left(0-\left(\tfrac{1}{2}\right)^3\right) = \frac{20^3}{75(8)} = \frac{20(400)}{600} = \frac{80}{6} = \boxed{\frac{40}{3}}$$

$\begin{array}{r} {}^{4}\\ 75\\ \times 8\\ \hline 600\end{array}$

Therefore the expected value of the distance the shot put would land from the origin is $\frac{40}{3}$ which makes sense because the area gets greater as r increases and $0 \le \frac{40}{3} \le 20$, so it is in the domain, therefore, the answer is a reasonable answer to the problem!

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $x^2+z^2=1$, $x^2+z^2=9$ and the planes $y=3$, $y=x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F}=\left\langle \frac{y}{x^2+y^2}, -\frac{x}{x^2+y^2}, 3z\right\rangle$ across $\mathcal{S}$.

First thing that comes to mind is divergence Theorem. The region would be much neater to integrate over after taking div.



Let's compute $\operatorname{div}\vec{F}$ to see if it truly makes our life easier.

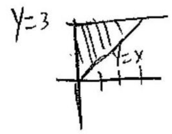
$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(-\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial z}(3z)$$

$$= \frac{-y(2x)}{(x^2+y^2)^2} - \frac{-x(2y)}{(x^2+y^2)^2} + 3 = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} + 3 = 3.$$

Therefore, we are going to want to apply the divergence theorem because it is much neater, we now need to find a W to integrate over! Cylindrical coordinates make the most sense because we are looking at cylinders.

Let $x = r\cos\theta$, $y=y$ and $z=r\sin\theta$, so that we have (r,θ,y) for our coordinates. $x^2+z^2=r^2=1$, so $r=1$ and $x^2+z^2=r^2=9$, so $r=3$, therefore r is bounded by 1 and 3, such that $1\le r\le 3$. As well we are looking at full circles in the cylinder so θ will be bounded by $0\le\theta\le 2\pi$. Now we need to find the bounds for y.



, so y is bounded by 3 on top and x on the bottom such that $x\le y\le 3$

and since we are looking at cylindrical coordinates, $x=r\cos\theta$, so $r\cos\theta\le y\le 3$.

$W=\{0\le\theta\le 2\pi,\ 1\le r\le 3,\ r\cos\theta\le y\le 3\}$

Therefore, we can write our integral as $\iiint_W \operatorname{div}\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S} = \int_0^{2\pi}\int_1^3\int_{r\cos\theta}^3 3\,r\,dy\,dr\,d\theta$

$$= \int_0^{2\pi}\int_1^3 3yr\Big]_{r\cos\theta}^3 dr\,d\theta = \int_0^{2\pi}\int_1^3 (9r - r^2\cos\theta)\,dr\,d\theta = \int_0^{2\pi} \frac{9}{2}r^2 - \frac{1}{3}r^3\cos\theta\Big]_1^3 d\theta =$$

$$= \int_0^{2\pi}\left(\frac{81}{2} - 9\cos\theta - \frac{9}{2} + \frac{1}{3}\cos\theta\right)d\theta = \int_0^{2\pi}\left(\frac{72}{2} - \frac{26}{3}\cos\theta\right)d\theta = \left(36\theta - \frac{26}{3}\sin\theta\right)\Big]_0^{2\pi} = 72\pi - 0 = \boxed{72\pi}.$$

Therefore, by applying the divergence theorem, we were able to compute the flux over the surface S.

7. (12 points) Let C be the curve

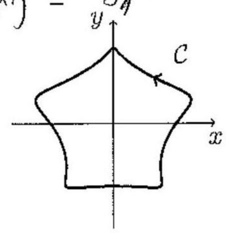
$\text{curl}_z\, F_2 = \frac{\partial(e^{e^y} + xe^{xy})}{\partial x} - \frac{\partial}{\partial y}(\cos x^3 + ye^{xy})$

$e^{xy} + y^2 e^{xy}$

$(x^2+y^2)e^{xy}$

$\int ye^{xy}\, dx$ $\quad y$

e^{xy}



Find $\oint_C \mathbf{F} \cdot dr$ where

↗ Try to apply Green's Theorem, compute curl$_z$ of F_1 and F_2 !

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \cos(x^3) + ye^{xy}, \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle.$$

(*Hint: Try writing* **F** *as a sum of two vector fields that we know how to integrate around* C.)

If we let $\vec{F} = \vec{F}_1 + \vec{F}_2$ such that $\vec{F}_1 = \left\langle \frac{-y}{x^2+y^2} + \cos(x^3), \frac{x}{x^2+y^2} + e^{e^y} \right\rangle$

and $\vec{F}_2 = \langle ye^{xy}, xe^{xy} \rangle$

We can easily show that $\cancel{\oint_C \vec{F}_1 \cdot d\vec{r} = 0}$ by applying Green's Theorem.

By Green's Theorem $\iint_D \text{curl}_z \vec{F}_1\, dA = \oint_C \vec{F}_1 \cdot d\vec{r}$, so We compute $\text{curl}\,\vec{F}_1 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} =$

$\frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2} + e^{e^y}\right) = \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$

$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2} + \cos(x^3)\right) = \frac{-((x^2+y^2) - y(2y))}{(x^2+y^2)^2} = \frac{-(x^2-y^2)}{(x^2+y^2)^2}$

$\text{curl}_z \vec{F}_1 = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} - \left(\frac{-(x^2-y^2)}{(x^2+y^2)^2}\right) = 0$, Therefore by Green's Theorem $\iint_D \text{curl}_z \vec{F}_1\, dA = \iint_D 0\, dA = 0 = \oint_C \vec{F}_1 \cdot d\vec{r}$

Now, for $\vec{F}_2$, We can find the Value over the curve by proving that $\vec{F}_2$ is conservative and thus $\oint_C \vec{F}_2 \cdot d\vec{r} = 0$ by the fundamental theorem of vector line integrals. $\vec{F}_2$ is Conservative if We are able to find a Potential function for $\vec{F}_2$. We can do this by finding the partial integral of each component of $\vec{F}_2$ and setting them equal to each other.

$f(x,y) = \int F_1\, dx = \int ye^{xy}\, dx = e^{xy} + f(y)$

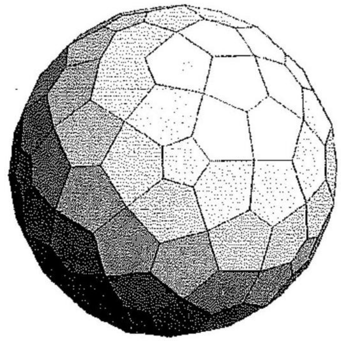
$f(x,y) = \int F_2\, dy = \int xe^{xy}\, dy = e^{xy} + g(x)$, and we set the two functions equal to each other

to get $e^{xy} + f(y) = e^{xy} + g(x)$ and for $g(x)=0$ and $f(y)=0$, we obtain the following potential function for any constant C, $f(x,y) = e^{xy} + C$.

Therefore, since we were able to obtain a potential function for $\vec{F}_2$, and we are trying to find $\oint_C \vec{F}_2 \cdot d\vec{r}$, where C is a closed curve, we can say $\oint_C \vec{F}_2 \cdot d\vec{r} = 0$ by the fundamental theorem of vector line integrals. Therefore, by the linearity property of the vector line integral we are able to say for $\vec{F} = \vec{F}_1 + \vec{F}_2$, $\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F}_1 \cdot d\vec{r} + \oint_C \vec{F}_2 \cdot d\vec{r} = 0$ because $\oint_C \vec{F}_1 \cdot d\vec{r} = 0$ by Green's theorem and $\oint_C \vec{F}_2 \cdot d\vec{r} = 0$ by the fundamental theorem of vector line integrals.

Therefore, $\boxed{\oint_C \vec{F} \cdot d\vec{r} = 0}$!

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{a}_j$ so that $\mathbf{a}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{a}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

a) It is known that $\|\vec{F}\|^2 = \vec{F} \cdot \vec{F}$, so we need to show that $\iint_S \vec{F} \cdot d\vec{S} = \vec{F} \cdot \vec{F}$.

We also know that $\iint_S \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S}_1 + \iint_{S_2} \vec{F} \cdot d\vec{S}_2 + \ldots + \iint_{S_K} \vec{F} \cdot d\vec{S}_K$. We can rewrite the integral $\iint_{S_1} \vec{F} \cdot d\vec{S}_1$ as $\iint_{S_1} \vec{F} \cdot \vec{a}_1\, dS = \vec{F} \cdot \vec{a}_1 \iint_{S_1} dS$, we can do this because $\vec{a}_1$ is a constant unit normal vector and $\vec{F} = \sum_{j=1}^{k} \text{Area}(S_j)\vec{a}_j$, which is a scalar multiplied by a constant unit normal vector, so if we dot two constant vectors together $\vec{a}_1$ and $\sum_{j=1}^{k} \underset{\text{constant}}{\text{area}(S_j)}\vec{a}_j$ we will obtain a constant scalar value. This allows us to pull $\vec{F} \cdot \vec{a}_1$ out of the integral as there is no dependency. Therefore, since we have $\iint_{S_1} \vec{F} \cdot d\vec{S} = \vec{F} \cdot \vec{a}_1 \iint_{S_1} dS$ we can actually say that $\iint_{S_1} dS_1 = \text{Area}(S_1)$ because it is an integral over the surface. Therefore, we obtain $\iint_{S_1} \vec{F} \cdot d\vec{S}_1 = \vec{F} \cdot \vec{a}_1 \text{Area}(S_1)$, which can be written as $\vec{F} \cdot \vec{N}_1$ by the definition above. ($\vec{N}_j = \text{Area}(S_j)\vec{a}_j$)

We can say that similarly for any planar surfaces, $\iint_{S_K} \vec{F} \cdot d\vec{S}_K = \vec{F} \cdot \vec{N}_K$. Therefore we can write the surface integral as $\iint_S \vec{F} \cdot d\vec{S} = \sum_{j=1}^{k} \vec{F} \cdot \vec{N}_K$, and since this summation has no dependency on $\vec{F}$, we can write $\iint_S \vec{F} \cdot d\vec{S} = \vec{F} \cdot \sum_{j=1}^{k} \vec{N}_j$

Also, by definition in the problem, $\vec{F} = \vec{N}_1 + \vec{N}_2 + \ldots + \vec{N}_K = \sum_{j=1}^{k} \vec{N}_j$. Therefore we can substitute $\vec{F}$ in for $\sum_{j=1}^{k} \vec{N}_j$ to get $\iint_S \vec{F} \cdot d\vec{S} = \vec{F} \cdot \vec{F} = \|\vec{F}\|^2$ $\square$ Q.E.D. :)

b) After proving $\|\vec{F}\|^2 = \iint_S \vec{F} \cdot d\vec{S}$, we can show that for S being a polyhedron as defined in the problem, $\vec{F} = 0$. Thinking about the problem logically at first, we know that $\vec{F} = 0$ because a polyhedron is similar to a sphere if k is large and the normal vector at every point on a sphere has a normal vector pointing in the opposite direction, so they will all cancel each other out if summed up. This same concept can be applied to the polyhedron because $\vec{F}$ is made up of normal vectors and not unit normals. The normal vectors are scaled in such a way that their values are proportional to the surface area of the planar surface S_i, so summing up all of those normal vectors will result in a value of 0.

Since $\vec{F} = \sum_{j=1}^{k} \vec{N}_j$ and $\vec{N}_j = \text{Area}(S_j)\vec{a}_j$, so $\vec{F} = \sum_{j=1}^{k} \text{area}(S_j)\vec{a}_j = 0$ because of the logic applied above.

As well we can show that $\iint_S \vec{F} \cdot d\vec{S} = 0$ because $\iint_S \vec{F} \cdot d\vec{S} = \vec{F} \cdot \sum_{j=1}^{k} \text{Area}(S_j)\vec{a}_j = 0$ because all of the normals are scaled by the area and will cancel each other out by pointing in opposite directions over the surface.