Math 32B - Lectures 3 & 4
Winter 2019
Final Exam
3/17/2019

Name: ____________________
SID: ____________________
TA Section: ____________________

Time Limit: 180 Minutes

Version ⟳

This exam contains 20 pages (including this cover page) and 8 problems. There are a total of 90 points available.

Check to see if any pages are missing. Enter your name, SID and TA Section at the top of this page.

You may **not** use your books, notes or a calculator on this exam.

Please **switch off your cell phone** and place it in your bag or pocket for the duration of the test.

- Attempt all questions.
- Write your solutions clearly, in full English sentences, using units where appropriate.
- You may write on both sides of each page.
- You may use scratch paper if required.

Mechanics formulas

- If $\mathcal{D}$ is a lamina with mass density $\delta(x, y)$ then
  - The mass is $M = \iint_{\mathcal{D}} \delta(x, y)\, dA$.
  - The y-moment is $M_y = \iint_{\mathcal{D}} x\, \delta(x, y)\, dA$.
  - The x-moment is $M_x = \iint_{\mathcal{D}} y\, \delta(x, y)\, dA$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iint_{\mathcal{D}} y^2\, \delta(x, y)\, dA$.
  - The moment of inertia about the y-axis is $I_y = \iint_{\mathcal{D}} x^2\, \delta(x, y)\, dA$.
  - The polar moment of inertia is $I_0 = \iint_{\mathcal{D}} (x^2 + y^2)\, \delta(x, y)\, dA$.
- If $\mathcal{W}$ is a solid with mass density $\delta(x, y, z)$ then
  - The mass is $M = \iiint_{\mathcal{W}} \delta(x, y, z)\, dV$.
  - The yz-moment is $M_{yz} = \iiint_{\mathcal{W}} x\, \delta(x, y, z)\, dV$.
  - The xz-moment is $M_{zx} = \iiint_{\mathcal{W}} y\, \delta(x, y, z)\, dV$.
  - The xy-moment is $M_{xy} = \iiint_{\mathcal{W}} z\, \delta(x, y, z)\, dV$.
  - The center of mass is $(x_{\text{CM}}, y_{\text{CM}}, z_{\text{CM}}) = \left(\frac{M_{yz}}{M}, \frac{M_{zx}}{M}, \frac{M_{xy}}{M}\right)$.
  - The moment of inertia about the x-axis is $I_x = \iiint_{\mathcal{W}} (y^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the y-axis is $I_y = \iiint_{\mathcal{W}} (x^2 + z^2)\, \delta(x, y, z)\, dV$.
  - The moment of inertia about the z-axis is $I_z = \iiint_{\mathcal{W}} (x^2 + y^2)\, \delta(x, y, z)\, dV$.

Probability formulas

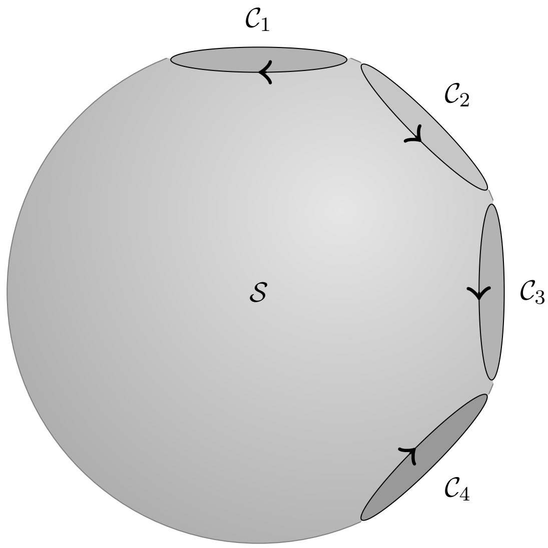
- If a continuous random variable X has probability density function $p_X(x)$ then
  - The total probability $\int_{-\infty}^{\infty} p_X(x)\,dx = 1$.
  - The probability that $a < X \le b$ is $\mathbb{P}[a < X \le b] = \int_a^b p_X(x)\,dx$.
  - If $f\colon \mathbb{R} \to \mathbb{R}$, the expected value of $f(X)$ is $\mathbb{E}[f(X)] = \int_{-\infty}^{\infty} f(x)\,p_X(x)\,dx$.
- If continuous random variables X, Y have joint probability density function $p_{X,Y}(x,y)$ then
  - The total probability $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_{X,Y}(x,y)\,dxdy = 1$
  - The probability that $(X,Y) \in \mathcal{D}$ is $\mathbb{P}[(X,Y) \in \mathcal{D}] = \iint_{\mathcal{D}} p_{X,Y}(x,y)\,dA$.
  - If $f\colon \mathbb{R}^2 \to \mathbb{R}$, the expected value of $f(X,Y)$ is $\mathbb{E}[f(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,p_{X,Y}(x,y)\,dxdy$.

1. (6 points) Find $\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}} \, dxdy$.

Solution: Using Fubini's Theorem we have

$$\int_0^2 \int_{y^2}^4 e^{x^{\frac{3}{2}}} \, dxdy = \int_0^4 \int_0^{\sqrt{x}} e^{x^{\frac{3}{2}}} \, dydx = \int_0^4 \sqrt{x} e^{x^{\frac{3}{2}}} \, dx = \frac{2}{3}\left(e^8 - 1\right).$$

2. (8 points) Let $\mathcal{S}$ be a part of the unit sphere $x^2 + y^2 + z^2 = 1$ oriented with outward pointing normal, with four holes bounded by the curves $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3, \mathcal{C}_4$ oriented as in the following picture:



Suppose that for a vector field $\mathbf{F}$ we have

$$\iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = 20, \qquad \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} = 305, \qquad \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} = 104, \qquad \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = 27.$$

Find $\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$.

Solution: By Stokes' Theorem we have

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} - \oint_{\mathcal{C}_3} \mathbf{F} \cdot d\mathbf{r} + \oint_{\mathcal{C}_4} \mathbf{F} \cdot d\mathbf{r} = \iint_{\mathcal{S}} \operatorname{curl} \mathbf{F} \cdot d\mathbf{S},$$

and hence

$$\oint_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 305 + 104 - 27 + 20 = 402.$$

3. (12 points) Let $\mathcal{C}$ be the part of the ellipse $x^2 + 4y^2 = 1$ bewteen $y = 0$ and $y = \frac{1}{2}x$ in the first quadrant. Find $\int_{\mathcal{C}} x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds$.

Solution: We parameterize $\mathcal{C}$ using $\mathbf{r}(t) = \langle \cos t, \frac{1}{2}\sin t\rangle$ for $0 \le t \le \frac{\pi}{4}$. We then have

$$\mathbf{r}'(t) = \langle -\sin t, \frac{1}{2}\cos t\rangle,$$

and hence

$$\|\mathbf{r}'(t)\| = \sqrt{\sin^2 t + \frac{1}{4}\cos^2 t}.$$

We then compute

$$\begin{aligned} \int_{\mathcal{C}} x\sqrt{\frac{1}{4}x^2 + 4y^2}\, ds &= \int_0^{\frac{\pi}{4}} \cos t \left(\frac{1}{4}\cos^2 t + \sin^2 t\right)\, dt \\ &= \int_0^{\frac{\pi}{4}} \cos t \left(\frac{1}{4} + \frac{3}{4}\sin^2 t\right)\, dt \\ &= \frac{3\sqrt{2}}{16}. \end{aligned}$$

4. (14 points) The solid $\mathcal{W}$ lies in the region where $x^2 + y^2 + z^2 \leq \frac{1}{100}$ and $\sqrt{3}z \leq -\sqrt{x^2 + y^2}$, where distance is measured in meters, and has constant density $\delta(x, y, z) = 5\,\mathrm{kg\,m^{-3}}$.
   (a) Write $\mathcal{W}$ using spherical coordinates.
   (b) Find the moment of inertia of $\mathcal{W}$ about the z-axis. *(Do not forget to use the correct units.)*

Solution:

(a) Using spherical coordinates

$$x = \rho\cos\theta\sin\phi, \qquad y = \rho\sin\theta\sin\phi, \qquad z = \rho\cos\phi,$$

we may write

$$\mathcal{W} = \{0 \leq \theta < 2\pi,\ \frac{2\pi}{3} \leq \phi \leq \pi,\ 0 \leq \rho \leq \frac{1}{10}\}.$$

(b) Switching to spherical coordinates we may compute

$$\begin{aligned} I_z &= \iiint_{\mathcal{W}} (x^2 + y^2)\delta(x, y, z)\,dV \\ &= \int_0^{2\pi}\int_{\frac{2\pi}{3}}^{\pi}\int_0^{\frac{1}{10}} 5\rho^4\sin^3\phi\,d\rho d\phi d\theta \\ &= \frac{\pi}{50000}\int_{\frac{2\pi}{3}}^{\pi} \sin\phi - \cos^2\phi\sin\phi\,d\phi \\ &= \frac{\pi}{50000}\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_{\phi=\frac{2\pi}{3}}^{\phi=\pi} \\ &= \frac{\pi}{240000}\,\mathrm{kg\,m^2} \end{aligned}$$

5. (14 points) A shot put throwing sector $\mathcal{D} \subset \mathbb{R}^2$ is bounded by the curves $x = 0$, $y = \sqrt{3}x$ and $x^2 + y^2 = 400$ in the first quadrant. On any given throw, the position at which my shot lands may be modelled by a pair of random variables (X, Y) with joint probability density

$$p_{X,Y}(x, y) = \begin{cases} \frac{3}{25} \frac{x^2 y}{(x^2 + y^2)^{\frac{3}{2}}} & \text{if } (x, y) \in \mathcal{D} \\ 0 & \text{otherwise,} \end{cases}$$

so that the distance I throw is $\sqrt{X^2 + Y^2}$. Find $\mathbb{E}[\sqrt{X^2 + Y^2}]$.

Solution: We may write the region $\mathcal{D}$ in polar coordinates as

$$\mathcal{D} = \left\{ \frac{\pi}{3} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 20 \right\}.$$

We then compute

$$\begin{aligned} \mathbb{E}[\sqrt{X^2 + Y^2}] &= \iint_{\mathbb{R}^2} \sqrt{x^2 + y^2}\, p_{X,Y}(x, y)\, dA \\ &= \frac{3}{25} \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \int_0^{20} r^2 \cos^2\theta \sin\theta \, dr d\theta \\ &= 320 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos^2\theta \sin\theta \, d\theta \\ &= \frac{40}{3}. \end{aligned}$$

6. (14 points) Let $\mathcal{S}$ be the boundary of the region $\mathcal{W}$ bounded by the cylinders $x^2 + z^2 = 1$, $x^2 + z^2 = 9$ and the planes $y = 3$, $y = x$ oriented with outward pointing normal. Find the flux of the vector field $\mathbf{F} = \left\langle \frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}, 3z \right\rangle$ across $\mathcal{S}$.

Solution: Taking cylindrical coordinates

$$x = r\cos\theta, \qquad y = y, \qquad z = r\sin\theta,$$

we may write the region $\mathcal{W}$ as

$$\mathcal{W} = \{1 \le r \le 3,\, 0 \le \theta < 2\pi,\, x \le y \le 3\}\,.$$

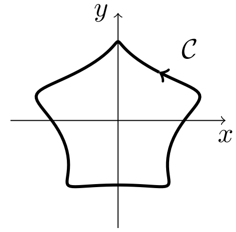
We also compute

$$\operatorname{div}\mathbf{F} = 3.$$

Applying the divergence theorem we then obtain

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \iiint_{\mathcal{W}} \operatorname{div}\mathbf{F}\, dV \\
&= \int_0^{2\pi}\int_1^3\int_{r\cos\theta}^3 3r\, dydrd\theta \\
&= \int_0^{2\pi}\int_1^3 9r - 3r^2\cos\theta\, drd\theta \\
&= \int_1^3 18\pi r\, dr \\
&= 72\pi.
\end{aligned}$$

7. (12 points) Let $\mathcal{C}$ be the curve



Find $\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y) = \left\langle -\frac{y}{x^2+y^2} + \cos(x^3) + ye^{xy},\, \frac{x}{x^2+y^2} + e^{e^y} + xe^{xy} \right\rangle.$$

(*Hint: Try writing* $\mathbf{F}$ *as a sum of two vector fields that we know how to integrate around* $\mathcal{C}$*.*)

Solution: Taking

$$\mathbf{F}_1(x,y) = \left\langle -\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle,$$

to be the vortex vector field and

$$\mathbf{F}_2(x,y) = \left\langle \cos(x^3) + ye^{xy},\, e^{e^y} + xe^{xy} \right\rangle,$$

we may write

$$\mathbf{F}(x,y) = \mathbf{F}_1(x,y) + \mathbf{F}_2(x,y).$$

From the theorem proved in class we have

$$\oint_{\mathcal{C}} \mathbf{F}_1 \cdot d\mathbf{r} = 2\pi.$$

Further, we may verify that for every $(x,y) \in \mathbb{R}^2$ we have

$$\text{curl}_z\, \mathbf{F}_2 = 0,$$

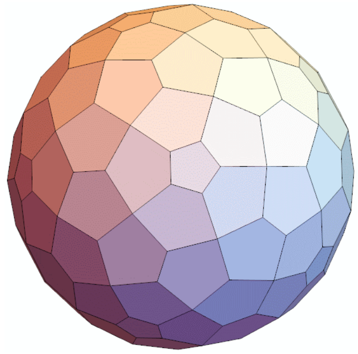
so by the Fundamental Theorem of Vector Line Integrals,

$$\oint_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r} = 0.$$

As a consequence,

$$\oint_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = \oint_{\mathcal{C}} \mathbf{F}_1 \cdot d\mathbf{r} + \oint_{\mathcal{C}} \mathbf{F}_2 \cdot d\mathbf{r} = 2\pi.$$

8. (10 points) Recall that a polyhedron is a solid bounded by several planar surfaces, for example



Let $\mathcal{W} \subset \mathbb{R}^3$ be a polyhedron with boundary $\mathcal{S}$ composed of k planar surfaces $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ so that

$$\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_k.$$

We orient $\mathcal{S}$ with the outward unit normal.

For each $j = 1, \ldots, k$ define the constant unit vector $\mathbf{a}_j$ so that $\mathbf{a}_j$ is equal to the outward unit normal to $\mathcal{S}$ on the surface $\mathcal{S}_j$. Define the constant vector $\mathbf{N}_j = \text{Area}(\mathcal{S}_j)\,\mathbf{a}_j$.

(a) Let $\mathbf{F} = \mathbf{N}_1 + \mathbf{N}_2 + \cdots + \mathbf{N}_k$. Show that

$$\|\mathbf{F}\|^2 = \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

(b) Using your answer to part (a), show that $\mathbf{F} = 0$.

Solution:

(a) We may write the surface integral as

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \sum_{j=1}^{k} \iint_{\mathcal{S}_j} \mathbf{F} \cdot \mathbf{n}\, dS = \mathbf{F} \cdot \left(\sum_{j=1}^{k} \mathbf{a}_j \iint_{\mathcal{S}_j} dS \right) = \mathbf{F} \cdot \left(\sum_{j=1}^{k} \text{Area}(\mathcal{S}_j)\,\mathbf{a}_j \right) = \|\mathbf{F}\|^2.$$

(b) Applying the Divergence Theorem we obtain

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iiint_{\mathcal{W}} \operatorname{div} \mathbf{F}\, dV = 0,$$

and hence $\|\mathbf{F}\|^2 = 0$.