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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Evaluate the following integral

$$\int\int_D x^2 + 2y^2 dxdy.$$

Here D is the finite region in $\mathbb{R}^2$ bounded by the lines $x + y = a, x + y = -a, x - y = b, x - y = -b$, where a and b are two positive constants.

$u = x+y \quad v = x-y$

$-a \le u \le a$
$-b \le v \le b$

$ac = \frac{1}{det} = det\left|\begin{matrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{matrix}\right| = det\left|\begin{matrix} 1 & 1 \\ 1 & -1 \end{matrix}\right|$

$u+v = 2x \quad x = \frac{u+v}{2} \quad u-v = 2y \quad y = \frac{u-v}{2}$

$\int_{-b}^{b}\int_{-a}^{a} \left(\frac{u+v}{2}\right)^2 + 2\left(\frac{u-v}{2}\right)^2 \left(\frac{1}{2}\right) du dv = \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a} \frac{1}{4}(u^2 + \ldots$

$\frac{u^3}{12} + \frac{u^2 v}{4} + uv^2 + \frac{u^3}{3} - u^2 v + uv^2 \Big|_{-a}^{a} dv = \frac{1}{2}\int_{-b}^{b} [\ldots$

$\cancel{\frac{5a^3}{12} + \frac{1}{4}av + \frac{2}{3}a^3 dv}$

(Check the page in back for work)

$= \frac{1}{2}\left(\cancel{\frac{5}{6}a^3(2b) + \frac{4}{3}a(b^3)2}\right)$

$= \boxed{\frac{1}{2}a^3 b + \frac{1}{2}ab^3}$

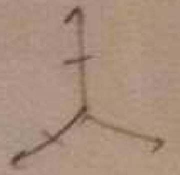
Problem 2. (4)

Find the line integral

$$\int_{C_1+C_2} yz\,dx + xz\,dy + xy\,dz, \qquad F = \langle yz, xz, xy \rangle$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,0,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

$C_1(t) \Rightarrow r_1(t) = (1-t)\langle 1,0,1\rangle + t\langle 1,1,0\rangle$



$r_1(t) = \langle 1, t, 1-t\rangle \quad 0 \le t \le 1$

$\int_{C_1+C_2} \vec{F}\,dr = \int_{C_1} F\,dr + \int_{C_2} F\,dr$

$r_1'(t) = \langle 0, 1, -1\rangle \quad \int_0^1 -2t+1\,dt = -t^2+t\Big|_0^1 = -1+1 = 0$

$F(r_1(t)) = \langle t-t^2, 1-t, t\rangle$

$F(r_1(t)) \cdot r_1'(t) = 0 + 1 - t - t$

$r_2(t) = (1-t)\langle 1,1,0\rangle + t\langle 0,1,1\rangle$

$r_2(t) = \langle 1-t, 1, t\rangle \quad 0 \le t \le 1$

$F(r_2(t)) = \langle t, t-t^2, 1-t\rangle$

$r_2'(t) = \langle -1, 0, 1\rangle$

$F(r_2(t)) \cdot r_2'(t) = -t + 0 + 1 - t$

$= -2t + 1$

2

$\int_0^1 -2t+1\,dt = -t^2+t\Big|_0^1 = -1+1 = 0$ ✓

$1 + 2 = 0$

so line integral is $\boxed{0}$. ✓

$= -4t+2\,dt = -2t^2+2t\Big|_0^1 = 0$ ✓

Problem 3. (4)

Let $\mathbf{F} = (x^3 + y)\mathbf{i} + (x + z - \sin y)\mathbf{j} + (z^2 + y + \cos z)\mathbf{k}$ defined on $\mathbb{R}^3$

(i) Decide if $\mathbf{F}$ is conservative.

(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.

(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t^2$, $y = t^4$, $z = t^6$, $0 \le t \le 1$ with the orientation given by the parametrization.

i) $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} \Rightarrow 1 = 1 ✓$ $\quad \frac{\partial}{\partial y}(x^3+y) = 1 \quad \frac{\partial}{\partial x}(x+z-\sin y) = 1 ✓$

✓ $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x} \Rightarrow 0 = 0 ✓$ $\quad \frac{\partial}{\partial z}(x^3+y) = 0 \quad \frac{\partial}{\partial x}(z^2+y+\cos z) = 0$ ✓

$\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} \Rightarrow 1 = 1 ✓$ $\quad \frac{\partial}{\partial z}(x+z-\sin y) = 1 \quad \frac{\partial}{\partial y}(z^2+y+\cos z) =$

since it is also defined on $\mathbb{R}^3$, with no holes, $\boxed{\text{F is conservative}}$ (simply connected on $\mathbb{R}^3$)

satisfies cross partials)

ii) $\frac{\partial}{\partial x} g(x,y,z) = x^3+y \Rightarrow \int (x^3+y)\,dx = \frac{x^4}{4} + xy + g(y,z)$

$\frac{\partial}{\partial y}\left(\frac{x^4}{4} + xy + g(y,z)\right) = x+z-\sin y$

✓ $x + g_y(y,z) = x + z - \sin y$

$g(y,z) = \int (z - \sin y)\,dy$ $\quad \frac{1}{4}+\frac{1}{3}$

$\boxed{= \frac{x^4}{4} + xy + yz + \cos y + \frac{z^3}{3} + \sin z + C}$ $\quad g(y,z) = yz + \cos y + g(z)$

$\frac{\partial}{\partial z}(yz + \cos y + g(z)) = z^2 + y + \cos z$

$y + g'(z) = z^2 + y + \cos z$

$g(z) = \int (z^2 + \cos z)$

$g(z) = \frac{z^3}{3} + \sin z$

iii) Since $\vec{F}$ is conservative,

$r(t) = \langle t^2, t^4, t^6 \rangle$ for $0 \le t \le 1$

$r(1) = \langle 1, 1, 1 \rangle$, $r(0) = \langle 0, 0, 0 \rangle$

✓ $\int_C F \cdot dr = V(1,1,1) - V(0,0,0)$ $\quad \frac{1}{4}+\frac{1}{3} = \frac{19}{12}$ (sic)

$\sin(1) + \cos(1) + \frac{19}{12}$ $\quad = \frac{1}{4} + 1 + 1 + \cos(1) + \frac{1}{3} + \sin(1) - (0 + 0 + 0 + 1 + 0 + 0)$

$\frac{28}{3}+\frac{12}{3}=\frac{40}{3}$ $\sin\phi = -\cos\phi$

$= -\cos\phi\Big|_{\pi}^{0}$

$= 1+1 = 2$

$f(x, y, z)$

Problem 4. (4)

Evaluate the surface integral $\iint_S (3x^2+4y^2+5z^2)\,dS$ where <u>S is the sphere</u> given by the equation $x^2+y^2+z^2=a^2$ where a is a positive constant.

$, \phi) = \langle a\sin\phi\cos\theta,\ a\sin\phi\sin\theta,\ a\cos\phi \rangle$

$a^2\sin\phi\ \hat{e}_\rho$ where $\hat{e}_\rho = \langle \sin\phi\cos\theta, \sin\theta\sin\phi, \cos\phi \rangle$

$= a^2\sin\phi$

$0 \le \theta \le 2\pi$

$0 \le \phi \le \pi$

$$\int_0^{2\pi}\int_0^{\pi} \left(3a^2\sin^2\phi\cos^2\theta + 4a^2\sin^2\phi\sin^2\theta + 5a^2\cos^2\phi\right) a^2\sin\phi\, d\phi\, d\theta$$

$\int_0^{2\pi}\cos^2\theta = \pi$ split up $\int_0^{2\pi}\sin^2\theta = \pi$

$$3a^4\sin^3\phi\cos^2\theta + 4a^4\sin^3\phi\sin^2\theta + 5a^4\sin\phi\cos^2\phi\, d\phi\, d\theta$$

$u = \cos\phi$ $du = -\sin\phi$

$(1-\cos^2\phi)\sin\phi + 4a^4\pi(1-\cos^2\phi)\sin\phi\, d\phi$ $+\ 2\pi\left(\frac{5}{3}\right)a^4\left(-\cos^3\phi\right)\Big|_0^{\pi}$

$= \frac{10\pi a^4}{3}(1+1)$

$\cos\phi\Big|_{\pi}^{0} + \frac{\cos^3\phi}{3}\Big|_0^{\pi}\Big) = \frac{28}{3}\left(2+\underbrace{\left(-\frac{1}{3}-\frac{1}{3}\right)}_{4/3}\right)$

$= \frac{28}{3}a^4\pi + \frac{20}{3}\pi a^4$

$= \frac{48}{3}\pi a^4 = \boxed{16\pi a^4}$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 + bv^2)$ with domain $D = \{(u,v) | u^2 + v^2 \leq 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{N}$ pointing to the negative z-direction (that is the z-component of $\mathbf{N}$ is negative). Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = \langle z, x^2, x^2 \rangle$.

(not negative)

$T_u = \langle 1, 0, 2au \rangle$

$T_v = \langle 0, 1, 2bv \rangle$

$$T_u \times T_v = \begin{vmatrix} 1 & 0 & 2au \\ 0 & 1 & 2bv \end{vmatrix} = \langle -2au, -2bv, 1 \rangle$$

$$\vec{N} = T_v \times T_u = \langle 2au, 2bv, -1 \rangle$$

(negative z)

$F(G(u,v)) = (bv^2 + au^2, u^2, u^2)$

$$F(G(u,v)) \cdot \vec{N} \Rightarrow \iint_{u^2+v^2 \leq 1} (2a^2u^3 + 2aubv^2 + 2u^2bv - u^2)\, du\, dv$$

b/c our region is symmetric over (both) the u & v planes, we know that $2a^2u^3$, $2aubv^2$, $2u^2bv$ will cancel by symmetry

Convert to polar...

$0 \leq r \leq 1$

$0 \leq \theta \leq 2\pi$

$u = r\cos\theta$

$v = r\sin\theta$

$\int_0^{2\pi} \cos^2\theta = \pi$

$$\int_0^{2\pi}\int_0^1 -(r^2\cos^2\theta \; r)\, dr\, d\theta = -\pi \int_0^1 r^3\, dr$$

$$= -\pi \left(\frac{1}{4}\right)$$

$$= \boxed{-\frac{\pi}{4}}$$