1. In lecture, we have used the "toy" or "model" theorem

$$\oint_{\text{boundary}} \text{function or vector field} = \int_{\text{region}} \text{"derivative" of function or vector field}$$

to explain the similarities among the Fundamental Theorem of Line Integrals, Green's Theorem, and Stokes' Theorem. In this question, you are asked to match the parts of this model theorem with those of the Divergence Theorem. (7 points)

(a) Write the equation that appears in the Divergence Theorem.

$$\iint_{\partial W} F \cdot dS = \iiint_{W} \operatorname{div}(F)\, dV$$

(b) What is the "region" that appears in the Divergence Theorem?
(Circle exactly one answer below.)

line OR curve OR surface OR 2-dimensional region OR 3-dimensional solid

(c) What is the "boundary" that appears in the Divergence Theorem?
(Circle exactly one answer below.)

line OR curve OR surface OR 2-dimensional region OR 3-dimensional solid

(d) What is the notion of "derivative" that appears in the Divergence Theorem?

the divergence

(e) What type of integral is the "big / bold integral" that appears in the Divergence Theorem?
(Circle exactly one answer below.)

single OR double OR triple OR line OR surface

(f) What type of integral is the "small / dotted integral" that appears in the Divergence Theorem?
(Circle exactly one answer below.)

single OR double OR triple OR line OR surface

2. (a) Describe with a picture *and* words a solid whose volume is given by

$$\int_0^{\pi}\int_0^{\pi}\int_1^{2}\rho^2\sin\phi\,\mathrm{d}\rho\,\mathrm{d}\phi\,\mathrm{d}\theta.$$

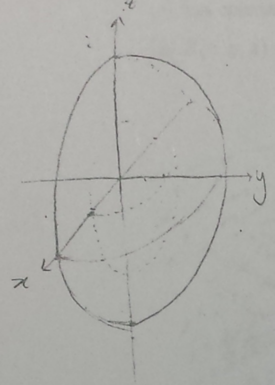
$1 \le \rho \le 2.$

$0 \le \phi \le \pi$ (vertically)

$0 \le \theta \le \pi.$ [illegible] right half

(3 points)

3



Take a sphere of radius 2, and split it in half (vertically). Take the right half of it, and scoop out a right hemisphere of radius 1 centered at the same point as the original sphere that we began with is (of radius 2).

6 (b) Parametrize the boundary of the solid from part (a). (6 points)

$\rho=2$: outer hemisphere: $r_1(\theta,\phi) = (2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi)$, $\theta: 0 \to \pi$, $\phi: 0 \to \pi$.

$\rho=1$: inner hemisphere: $r_2(\theta,\phi) = (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi)$, $\theta: 0 \to \pi$, $\phi: 0 \to \pi$.

$1<\rho<2$: flat ⊙ surface: $r_3(r,\phi) = (r\cos\phi,\ r\sin\phi)$, $r: 1 \to 2$, $\phi: 0 \to 2\pi$ ✶

↑ 0

$$S(\rho,\theta,\phi) = \begin{cases} \rho=1: & (\cos\theta\sin\phi,\ \sin\theta\sin\phi,\ \cos\phi) \\ \rho=2: & (2\cos\theta\sin\phi,\ 2\sin\theta\sin\phi,\ 2\cos\phi) \\ 1<\rho<2: & (\rho\cos(2\phi),\ 0,\ \rho\sin(2\phi)) \end{cases}$$ ok

where ρ goes from 1 to 2,

θ goes from 0 to π,

ϕ goes from 0 to π.

3. Match each plot with one of the following equations or properties. **Justify your answers.** (2 points each)

(a) has divergence zero no net flow

(b) $\mathbf{F}(x,y,z) = (1,2,z)$

(c) has constant negative divergence

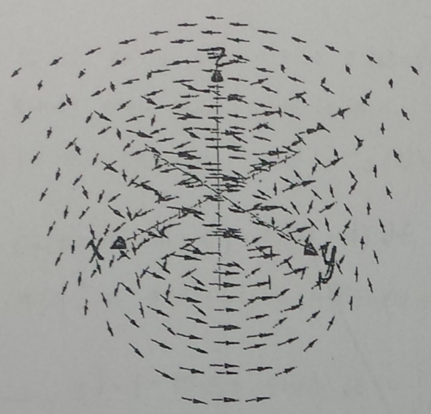
(d) $\mathbf{F}(x,y,z) = (x,y,z)$

(e) has constant curl pointing in the direction $(0,-1,0)$

(f) has constant curl pointing in the direction $(1,0,1)$

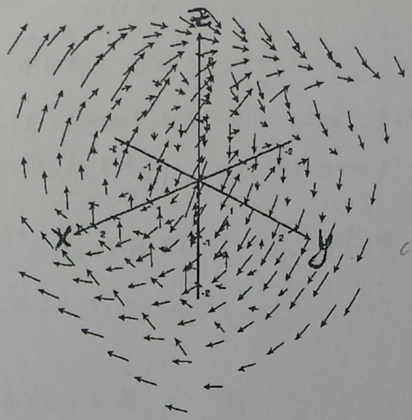
(g) $\mathbf{F}(x,y,z) = (0,y,1)$ only depends on y.

(h) has positive surface integral over the unit sphere outward.



(I) (a)

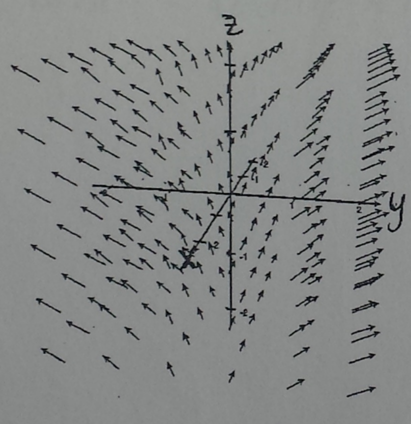
(a) The direction of a vector at each point is tangential to a sphere. This means there is no net flow into/out of the sphere. Therefore, the divergence is 0.



(II) (e).

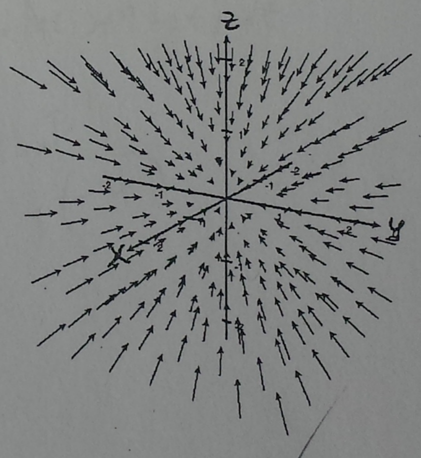
curl in negative y

(e) The curl of this vector field is in negative y direction. (right hand rule). (e) is the only option.



(III) (g)

(g). The vector value at each point depends only on y. x and z values have no effect on it. Therefore, $F(x,y,z) = (0,y,1)$. → This is the only choice that depends solely on y.



no curl.
-div.
depends on x, y, z

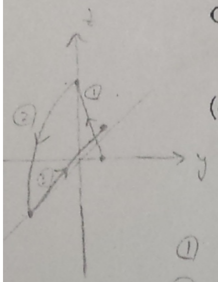
(IV) (c)

There is no fluid creation/destruction as we expand our sphere. (As we get further from the origin). (h) is not true because the flux is in the opposite direction as the normal of the unit sphere, which means the surface integral should be negative.

4. Let $\mathbf{F}(x, y, z) = (yz\cos(xy), xz\cos(xy) - z, \sin(xy) - y)$. Let C be the path formed by first following the line segment from $(0, 1, 0)$ to $(0, 0, 4)$; then following a quarter circle in the xz-plane centered at the origin from $(0, 0, 4)$ to $(4, 0, 0)$; and then following a line segment from $(4, 0, 0)$ to $(1, 1, 1)$.

Compute

$$\int_C \mathbf{F} \cdot d\mathbf{s}.$$



(8 points) $\int_C F \cdot ds = \int_{(0,1,0) \text{ to } (0,0,4)}^{(1)} F \cdot ds + \int_{(0,0,4) \text{ to } (4,0,0)}^{(2)} F \cdot ds + \int_{(4,0,0) \text{ to } (1,1,1)}^{(3)} F \cdot ds.$ $4(1-t)$

① line from $(0,1,0)$ to $(0,0,4)$: $c_1(t) = (0, t, \overset{4-4t}{4(1-t)})$ where $t: 1 \to 0$.

② quarter circle from $(0,0,4)$ to $(4,0,0)$: $c_2(t) = (4\cos\theta, 0, 4\sin\theta)$ where $t: \pi/2 \to \pi$.

③ line from $(4,0,0)$ to $(1,1,1)$: $c_3(t) = (\underset{4-3t}{4(1-t)+t}, t, t)$ where $t: 0 \to 1$.

① $\int_1^0 (\sim, 0-4+4t, 0-t) \cdot (0, 1, -4)\, dt = \int_1^0 (-4+4t+4t)\, dt = \int_1^0 8t-4\, dt$
$= |4t^2-4t|_1^0 = 4|t^2-t|_1^0 = 4((0)-(1-1)) = 4(0-0) = 0.$

② $\int_{\pi/2}^{\pi} (0, \sim, 0) \cdot (-4\sin\theta, 0, 4\cos\theta)\, dt = \int_{\pi/2}^{\pi} 0+0+0\, dt = 0.$

③ $\int_0^1 (t^2\cos(4t-3t^2), (4t-3t^2)\cos(4t-3t^2) - t, \sin(4t-3t^2) - t) \cdot (-3, 1, 1)\, dt$

$= \int_0^1 (-3t^2\cos(4t-3t^2) + (4t-3t^2)\cos(4t-3t^2) - t + \sin(4t-3t^2) - t)\, dt$

$= \int_0^1 (-6t^2\cos(4t-3t^2) + 4t\cos(4t-3t^2) + \sin(4t-3t^2) - 2t)\, dt$

FIX!

(-4)

$\langle 0,0,0\rangle \neq 0$..

2D function $\downarrow \nabla$ vf $\downarrow$ curl function

3D function $\downarrow \nabla$ vf $\downarrow$ curl vf $\downarrow$ div function

5. For each of the following statements, circle TRUE or FALSE. (2 points each)

(a) For any function f, $\text{curl}(\nabla f) = 0$.

~~In 3D, curl(∇f) ... a vector field, not function.~~

TRUE FALSE

(b) For any three-dimensional vector field $\mathbf{F}(x, y, z)$, $\text{curl}(\underbrace{\text{curl}\,\mathbf{F}}_{\text{vf.}}) = 0$.

TRUE FALSE

(c) For any function $f(x, y, z)$, $\text{div}(\nabla f) = 0$.

TRUE FALSE

(d) If a three-dimensional vector field $\mathbf{F}$ has a vector potential, then $\text{div}(\mathbf{F}) = 0$. div(curl G) = 0.

F = curl G.

TRUE FALSE

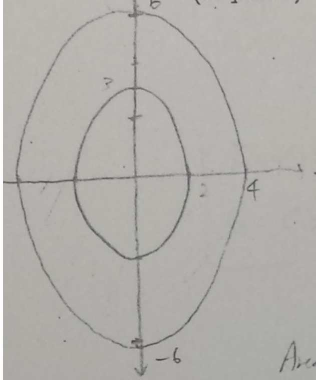
has a potential function.

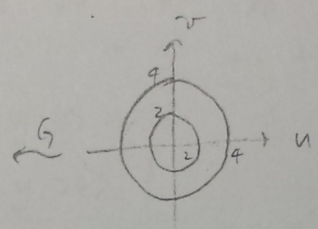
(e) If a three-dimensional vector field $\mathbf{F}$ is conservative, then $\mathbf{F}$ has a vector potential.

TRUE FALSE

2, 3.　　　4, 6.

6. Use a double integral to compute the area between the ellipses $\frac{x^2}{4}+\frac{y^2}{9}=1$ and $\frac{x^2}{16}+\frac{y^2}{36}=1$. (10 points)





$G(u,v) = (u, \frac{3}{2}v)$

$\mathrm{Jac}(G) = \begin{vmatrix} 1 & 0 \\ 0 & \frac{3}{2} \end{vmatrix} = \frac{3}{2}.$

Area b/w 2 ellipses

$= \iint_{D_0} \frac{3}{2}\, dA$　　where D_0 is the region b/w two circles in the

$= \frac{3}{2} \iint_{D_0} dA$

$= \frac{3}{2}$ (Area b/w two circles in the uv-plane)

$= \frac{3}{2}(16\pi - 4\pi)$

$= \frac{3}{\cancel{2}} \cdot \frac{\cancel{12}\pi}{6}$

$= \boxed{18\pi}$

2

7. (a) Give an example of two distinct 3-dimensional vector fields $\mathbf{F}$ and $\mathbf{G}$ with $\operatorname{div}\mathbf{F} = \operatorname{div}\mathbf{G}$.
(4 points)

$$\frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} = \frac{dG_1}{dx} + \frac{dG_2}{dy} + \frac{dG_3}{dz}$$

$$2 + 3 + 1 = 3 + 1 + 2$$

$$\boxed{\begin{aligned} F &= \langle 2x, 3y, z \rangle, \\ G &= \langle 3x, y, 2z \rangle \end{aligned}}$$

4

(b) Show that for every function $f : \mathbb{R}^3 \to \mathbb{R}$, there is a 3-dimensional vector field $\mathbf{F}$ with $\operatorname{div}\mathbf{F} = f$.
(8 points) 3D function. 3D vf div F: f

If $\operatorname{div} F = f$ where F is a 3D vector field,

then $\frac{dF_1}{dx} + \frac{dF_2}{dy} + \frac{dF_3}{dz} = f(x, y, z)$. ~ ①

Because we only have to show that there "exists" a 3D vector field F such that ① is true, we can set $\frac{dF_2}{dy}$ and $\frac{dF_3}{dz}$ as 0.

Then, $\frac{dF_1}{dx} = f(x, y, z)$.

Now, $f(x, y, z)$ only consists of x's, y's, z's, and possibly constants.

8

Then, we can simply integrate f with respect to x, treating y, z, (and of course constants) as constants.

For example, let $f(x, y, z)$ be expressed as A.

A could be any 3dimensional function.

Then, all we need to do is treat everything except x as a constant, and do $F_1 = \int A\, dx$.

Then, we have a F_1 such that $F_1 = \int A\, dx$, hence $\frac{dF_1}{dx} = A = f(x, y, z)$ where F_2 and F_3 are simply 0, or constants.

Therefore, for every function $f: \mathbb{R}^3 \to \mathbb{R}$, there exists a 3D vector field F with $\operatorname{div} F = f$. ■

This explanation is too long-winded.

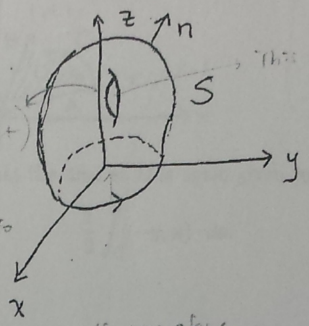
8. Let

$$\mathbf{F}(x,y,z) = \left(\frac{y-z}{\sqrt{x^2+y^2+z^2}}, \frac{z-x}{\sqrt{x^2+y^2+z^2}}, \frac{x-y}{\sqrt{x^2+y^2+z^2}} \right).$$

Compute the flux of $\mathbf{F}$ through the surface S pictured below, with upward-pointing normal vector (as shown). You may assume that:

- the boundary of S is the unit circle in the xy-plane;
- $\mathbf{F}$ has a vector potential.

(12 points)



This is NOT a boundary.

The little "hole" in the middle (() this part) is NOT a boundary.

By Stokes' Theorem, $\iint_S F \cdot dS$ is equal to $\iint_{S_2} F \cdot dS$ where S_2 has the same boundary as S does, which is a unit circle in the xy-plane.

One simple possible S_2 is a surface (a region) inside the unit circle.

Then, $\iint_S F \cdot dS = \iint_{S_2} F \cdot dS = \int_{\partial S_2}$ vector potential of $F \cdot ds$.

This is more simple.

$\iint_{S_2} F \cdot dS$: S_2 can be parametrized by $r(r,\theta) = \langle r\cos\theta, r\sin\theta, 0\rangle$ where $r: 0 \to 1$, $\theta: 0 \to 2\pi$.

$$n = C_r \times C_\theta = (\cos\theta, \sin\theta, 0) \times (-r\sin\theta, r\cos\theta, 0) = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r\cos^2\theta + r\sin^2\theta \rangle$$

$$= \langle 0, 0, r \rangle.$$

The normal should be upward-pointing in order to match S's boundary orientation, so we use $n = \langle 0, 0, r\rangle$.

$$F \cdot n = \left\langle \frac{r\sin\theta}{1}, \frac{-r\cos\theta}{1}, \frac{r\cos\theta - r\sin\theta}{1} \right\rangle \cdot \langle 0, 0, r \rangle$$

4

$$= 0 + 0 + r^2(\cos\theta - \sin\theta)$$

$$\iint_{S_2} F \cdot dS = \int_0^1 \int_0^{2\pi} r^2(\cos\theta - \sin\theta)\, d\theta\, dr = \int_0^1 r^2\, dr \cdot \int_0^{2\pi} \cos\theta - \sin\theta\, d\theta$$

$$= \left| \frac{1}{3} r^3 \right|_0^1 \cdot \left| \sin\theta + \cos\theta \right|_0^{2\pi}$$

$$= \frac{1}{3}(1) \cdot \left((0+1) - (0+1) \right)$$

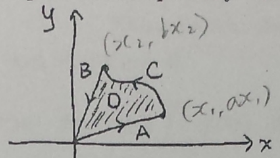
$$= 1 \cdot (1-1)$$

$$= \boxed{0}$$

9. In the homework, you used Green's Theorem to show that if C is a simple closed curve in $\mathbb{R}^2$, the area enclosed by C is given by

$$\frac{1}{2}\int_C (-y, x) \cdot \mathrm{ds}.$$

For this problem, suppose that C is not closed, but has the form shown below. Let A and B be lines through the origin with slopes a and b respectively. Let D be the region enclosed by A, B, and C.



If $F = (-y, x)$, then curl $F = 1-(-1) = 2$

Then; $\int_C (-y, x)\cdot ds = \iint_{\substack{\text{region}\\ \text{enclosed by}\\ C}} 2 \cdot dA = 2 \iint_{\substack{\text{region}\\ \text{enclosed by}\\ C}} 1\, dA$

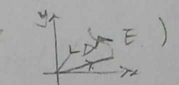
$= 2 \cdot$ Area enclosed by C

$\therefore$ (Area enclosed by C) $= \frac{1}{2}\int_C (-y, x)\cdot ds$

Use Green's Theorem to show that the area of D is again given by

$$\frac{1}{2}\int_C (-y, x) \cdot \mathrm{ds}.$$

(12 points)

Let E be a curve such that E is a combination of A, C, and B, maintaining their original orientation.
(E would look like ). Then, since E is a simple (no self intersections) and closed (same beginning and end points) curve, by Green's Theorem (shown above in the right corner) the area enclosed by E is $\frac{1}{2}\int_E (-y, x)\cdot ds$.

Then, we need to show that $\frac{1}{2}\int_E (-y, x)\cdot ds = \frac{1}{2}\int_C (-y, x)\cdot ds$.

$\frac{1}{2}\int_E (-y, x)\cdot ds = \frac{1}{2}\int_A (-y, x)\cdot ds + \frac{1}{2}\int_C (-y, x)\cdot ds + \frac{1}{2}\int_B (-y, x)\cdot ds.$

) Consider : $\frac{1}{2}\int_A (-y, x)\cdot ds$. Line A can be parametrized by $c_1(t) = (t, at)$ where t goes from 0 to an arbitrary x_1.

Then $\frac{1}{2}\int_A (-y, x)\cdot ds = \frac{1}{2}\int_0^{x_1} (-at, t)\cdot(1, a)\, dt = \frac{1}{2}\int_0^{x_1} (-at + at)\, dt = \frac{1}{2}\int_0^{x_1} 0\, dt = 0.$

consider : $\frac{1}{2}\int_B (-y, x)\cdot ds$. Line B can be parametrized by $c_2(t) = (t, bt)$ where t goes from 0 to an arbitrary x_2.

Then $\frac{1}{2}\int_B (-y, x)\cdot ds = \frac{1}{2}\int_0^{x_2} (-bt, t)\cdot(1, b)\, dt = \frac{1}{2}\int_0^{x_2} (-bt + bt)\, dt = \frac{1}{2}\int_0^{x_2} 0\, dt = 0.$

Then, $\frac{1}{2}\int_E (-y, x)\cdot ds = \frac{1}{2}\int_A (-y, x)\cdot ds + \frac{1}{2}\int_C (-y, x)\cdot ds + \frac{1}{2}\int_B (-y, x)\cdot ds$

$= 0 + \frac{1}{2}\int_C (-y, x)\cdot ds + 0$

$= \frac{1}{2}\int_C (-y, x)\cdot ds.$

Therefore, the area of D is $\iint_{\substack{\text{Region enclosed}\\ \text{by } E}} 1\, dA = \frac{1}{2}\int_E (-y, x)\cdot ds = \boxed{\frac{1}{2}\int_C (-y, x)\cdot ds.}$ ■

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10. Let S be a surface in $\mathbb{R}^3$ without boundary and W be the solid it bounds. Let f be a function that is defined and has continuous partial derivatives of every order on an open region containing W. Suppose that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

and that

$$f(x, y, z) = 0 \text{ on the surface } S.$$

Show that $f(x, y, z) = 0$ on W. (12 points)

Hint: Apply the divergence theorem to $\iint_S (f\nabla f) \cdot d\mathbf{S}$.

Warning: Do not spend too long on this problem unless you have finished the rest of your exam.

$$\iint_S (f\nabla f)\cdot dS = \iiint_W \operatorname{div}(f\nabla f)\cdot dV$$

$$\operatorname{div}(f\nabla f) = \operatorname{div}\left(f\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)\right) = \operatorname{div}\left\langle f\frac{\partial f}{\partial x}, f\frac{\partial f}{\partial y}, f\frac{\partial f}{\partial z}\right\rangle$$

$$= \frac{\partial}{\partial x}\left(f\frac{\partial f}{\partial x}\right) + \frac{\partial}{\partial y}\left(f\frac{\partial f}{\partial y}\right) + \frac{\partial}{\partial z}\left(f\frac{\partial f}{\partial z}\right)$$

$$= \left(\frac{\partial f}{\partial x}\cdot\frac{\partial f}{\partial x} + f\frac{\partial^2 f}{\partial x^2}\right) + \left(\frac{\partial f}{\partial y}\cdot\frac{\partial f}{\partial y} + f\frac{\partial^2 f}{\partial y^2}\right) + \left(\frac{\partial f}{\partial z}\cdot\frac{\partial f}{\partial z} + f\frac{\partial^2 f}{\partial z^2}\right)$$

$$= f\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$

$$= 0 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2.$$

$\iint_S (f\nabla f)\cdot dS = \iint 0 \, dS$ because $f(x,y,z) = 0$ on the surface S.

Then, $0 = \iiint_W \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 dV.$

This means $\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2 = 0.$

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Because all three terms are squares, in order for the sum to be 0, each square has to be 0.

Then, $\left(\frac{\partial f}{\partial x}\right)^2 = 0 \longrightarrow \frac{\partial f}{\partial x} = 0,$

$\left(\frac{\partial f}{\partial y}\right)^2 = 0 \longrightarrow \frac{\partial f}{\partial y} = 0.$

$\left(\frac{\partial f}{\partial z}\right)^2 = 0 \longrightarrow \frac{\partial f}{\partial z} = 0.$

$\because$ $f(x,y,z)$ consists of x's, y's, z's and possibly constants, and in order for all 3 partial derivatives to be 0, $f(x,y,z)$ must not have any x, y, z's in it.

Then $f(x,y,z)$ is a constant, which means its value does NOT change.

Since $f(x,y,z) = 0$ on the surface S, $f(x,y,z)$ must be 0 on W as well

$\boxed{\text{Therefore, } f(x,y,z) = 0 \text{ on } W.}$

$\square$