Course 32B - Midterm2

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February 19, 2014

25/25

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Instructions: Show all work to receive full credit. Feel free to use the back of each paper but **please indicate that you have done so**. No calculator or any document allowed. Justify all results. **Box your results**, clarity will be appreciated!

Problem		Possible	Score	Comment
1	1.1	2	2	
	1.2	2	2	
	1.3	4	4	Integral setup: 2pt + final answer: 2pt
2	2.1	2	2	
	2.2	4	4	1pt for each boundary
	2.3	2	2	1pt for the graph + 1pt for the definition of S
	2.4	2	2	
	2.5	4	4	Integral setup: 2pt + final answer: 2pt
3		2	2	Integral setup: 1pt + final answer: 1pt
Presentation		1	1	
Total		25	25	

1 Let the solid S lying above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = z$.

2

1.1 Sketch the profile of this solid in the xz-plane.

$x^2+y^2+z^2-z=0 \rightarrow x^2+y^2+z^2-z+\frac{1}{4}=\frac{1}{4}$

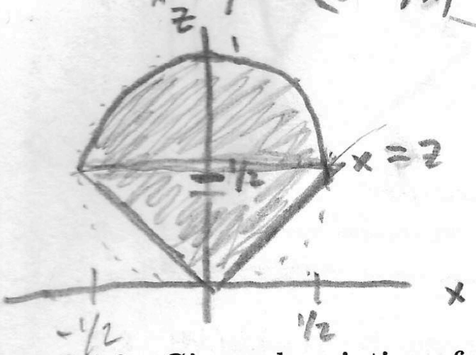
$x^2+y^2+(z-1/2)^2 = 1/4$

$(z^2-z+1/4) = (z-1/2)^2$

$p = 1/2$

$z = \sqrt{x^2+0} = \sqrt{x^2} = x$

$z = x$



2

1.2 Give a description of S in terms of spherical coordinates.

$0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi/4 \qquad 0 \le p \le \cos\phi$

$x = z$

$\downarrow$

$\phi = \pi/4$

$p^2\cos^2\theta\sin^2\phi + p^2\sin^2\theta\sin^2\phi + p^2\cos^2\phi - p\cos\phi + \cancel{1/4} = \cancel{1/4}$

$p^2\sin^2\phi\,(\cancel{\cos^2\theta+\sin^2\theta}) + p^2\cos^2\phi - p\cos\phi = 0$

$p^2(\sin^2\phi+\cos^2\phi) - p\cos\phi = 0 \rightarrow p^{\cancel{2}} = \cancel{p}\cos\phi$

$p = \cos\phi$

$\boxed{\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\cos\phi} p^2\sin\phi \, dp\,d\phi\,d\theta}$

$\boxed{0 \le \theta \le 2\pi \quad 0 \le p \le \cos\phi \quad 0 \le \phi \le \pi/4}$

1.3 Evaluate the volume of S by using the integral in spherical coordinates.

$V = \frac{1}{3}\int_0^{2\pi} d\theta \int_0^{\pi/4} p^3 \sin\phi \Big|_0^{\cos\phi} d\phi$

$= \frac{1}{3}(2\pi)\int_0^{\pi/4} \cos^3\phi \sin\phi \, d\phi$

$= \frac{2\pi}{3}\left[-\frac{\cos^4\phi}{4}\Big|_0^{\pi/4}\right]$

$\frac{d}{d\phi}\left(\frac{-\cos^4\phi}{4\sin\phi}\right)$

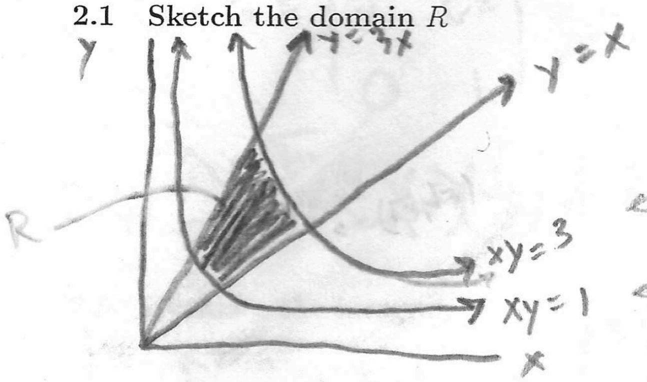
$\rightarrow (\cancel{-4}\cos^4\phi)(\cancel{-\sin\phi})$

$= -\cos^4\phi$

$= \left(-\frac{2\pi}{12}\right)\left[(\sqrt{2}/2)^4 - 1\right] = \frac{-\pi}{6}\left(\frac{4}{16}-1\right) = \frac{-\pi}{\cancel{6}}\left(\frac{-\cancel{12}}{8\cancel{16}}\right) = \boxed{\frac{\pi}{8}}$

2 We want to evaluate the integral $\int\int_R xy dA$ where R is the region in the first quadrant bounded by the lines $y = x$, $y = 3x$ and the hyperbolas $xy = 1$ and $xy = 3$.

2.1 Sketch the domain R



$xy = 3$

$y = \frac{3}{x}$

$xy = 1$

$y = \frac{1}{x}$

2/2

2.2 By using the change of variable $x = \frac{u}{v}$ and $y = v$, give for each boundary of R, the equations of the corresponding boundaries of the transformed domain S.

$x = \frac{u}{v}$ $y = v$

$y = x$:

$v = \frac{u}{v} \to v^2 = u \to v = \sqrt{u}$

$y = 3x$:

$v = \frac{3u}{v} \to v^2 = 3u \to v = \sqrt{3u}$

$\boxed{\sqrt{u} \leq v \leq \sqrt{3u}}$

4/4

$xy = 1$:

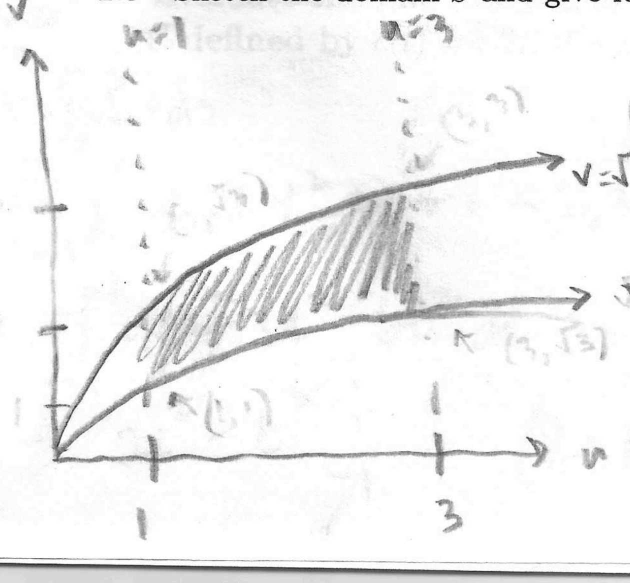
$\left(\frac{u}{v}\right)v = 1 \to u = 1$

$xy = 3$:

$\left(\frac{u}{v}\right)v = 3 \to u = 3$

$\boxed{1 \leq u \leq 3}$

2.3 Sketch the domain S and give its definition.



$\{S : g(u,v) : 1 \leq u \leq 3 ; \sqrt{u} \leq v \leq \sqrt{3u}$

2/2

2.4 Give the expression of the integral over S by using the previous change of variables.

$1 \le u \le 3$ $\quad \sqrt{u} \le v \le \sqrt{3u}$ $\quad x = u/v$ $\quad y = v$

$$\text{Jac } G = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$$

$$\iint_D xy \, dA$$

$$= \int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \left(\frac{u}{\cancel{v}}\right)(\cancel{v})\left(\frac{1}{v}\right) dv\,du = \boxed{\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, dv\,du}$$

+2

2.5 Evaluate the integral.

$$\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} \frac{u}{v} \, dv\,du = \int_1^3 \left[u \ln v \Big|_{\sqrt{u}}^{\sqrt{3u}} \right] du$$

$$= \int_1^3 u \left[\ln \sqrt{3u} - \ln \sqrt{u} \right] du$$

$$= \int_1^3 u \left[\ln \sqrt{3} + \cancel{\ln \sqrt{u}} - \cancel{\ln \sqrt{u}} \right] du$$

$$= \int_1^3 u \ln \sqrt{3} \, du = \frac{\ln \sqrt{3}}{2} \left[u^2 \Big|_1^3 \right] = \frac{\ln \sqrt{3}}{2} (9 - 1) = \frac{8 \ln \sqrt{3}}{2}$$

$$= \boxed{4 \ln \sqrt{3}}$$

+4

3 Evaluate the line integral of $\mathbf{F}(x, y, z) = < x, -z, y >$ along the curve $\mathcal{C}$ defined by $c(t) = < 2t, 3t, -t^2 >$ for $-1 \leqslant t \leqslant 1$.

$$= \int_C F \cdot ds \qquad c'(t) = < 2, 3, -2t >$$

$$= \int_{-1}^{1} < 2t, t^2, 3t > \cdot < 2, 3, -2t > dt \qquad -1 \le t \le 1$$

$$= \int_{-1}^{1} 4t + 3t^2 - 6t^2 \, dt = \int_{-1}^{1} 4t - 3t^2 \, dt$$

+2

$$= 2t^2 - t^3 \Big|_{-1}^{1} = (2-1) - (2+1) = 1 - 3 = \boxed{-2}$$