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MATH 32B Midterm II, Winter 2019

Name:

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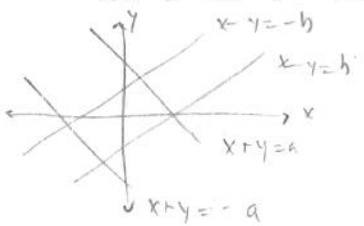
Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Evaluate the following integral

$$\int\int_{D} x^2 + 2y^2 + 3x\, dxdy.$$

Here D is the finite region in $\mathbb{R}^2$ bounded by the lines $x + y = a, x + y = -a, x - y = b, x - y = -b$, where a and b are two positive constants.



$u = x + y \qquad v = x - y$

$x = \frac{(x+y)+(x-y)}{2} = \frac{u+v}{2} \qquad y = \frac{(x+y)-(x-y)}{2} = \frac{u-v}{2}$

$G(u,v) = \langle \frac{u+v}{2}, \frac{u-v}{2} \rangle$

$-a \le x+y \le a, \; -b \le x-y \le b \Rightarrow -a \le u \le a, \; -b \le v \le b$

$D:$

$D_0: \{(u,v): -a \le u \le a, -b \le v \le b\}$

$$\iint_D (x^2+2y^2+3x)\,dx\,dy = \iint_{D_0} \left(\left(\frac{u+v}{2}\right)^2 + 2\left(\frac{u-v}{2}\right)^2 + 3\left(\frac{u+v}{2}\right)\right) |\mathrm{Jac}(G)|\, du\, dv$$

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

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$$\iint_D (x^2+2y^2+3x)\,dx\,dy = \int_{-b}^{b}\int_{-a}^{a} \frac{u^2+2uv+v^2+2u^2-4uv+2v^2+6u+6v}{4} \left|-\frac{1}{2}\right| du\, dv$$

$$= \int_{-b}^{b}\int_{-a}^{a} \frac{3u^2+6u+3v^2+6v-2uv}{8}\, du\, dv = \frac{1}{8}\int_{-b}^{b} \left(u^3+3u^2+u(3v^2+6v)-u^2v\right)\Big|_{u=-a}^{a} dv$$

$$= \frac{1}{8}\int_{-b}^{b} \left(a^3 \cancel{+3a^2} + a(3v^2+6v) \cancel{-a^2v} - (-a)^3 \cancel{-3(-a)^2} - (-a)(3v^2+6v) \cancel{+(-a)^2v}\right) dv$$

$$= \frac{1}{4}\int_{-b}^{b} \left(a^3 + a(3v^2+6v)\right) dv = \frac{1}{4}\left(a^3v + a(v^3+3v^2)\right)\Big|_{v=-b}^{b} = \frac{1}{4}\left(a^3b + ab^3 \cancel{+3ab^2} - a^3(-b) - a(-b)^3 \cancel{-3a(-b)^2}\right)$$

$$= \boxed{\frac{1}{2}\left(a^3b + ab^3\right)}$$

Problem 2. (4)

Let $\mathbf{F} = < y^2, yz, -xy >$. Find the line integral $\int_{C=C_1-C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,1,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

$C_1 = \overline{PQ}: \ \vec{r}_1(t) = <1, 1, 1-t>, \quad 0 \le t \le 1 \quad \vec{r}_1{}'(t) = <0, 0, -1>$

$C_2 = \overline{QR}: \ \vec{r}_2(t) = <1-t, 1, t>, \quad 0 \le t \le 1 \quad \vec{r}_2{}'(t) = <-1, 0, 1>$

$\vec{F}(\vec{r}_1(t)) = <1^2, 1(1-t), -1(1)> = <1, 1-t, -1>$

$\vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t) = 0 + 0 + (-1)(-1) = 1$

$\vec{F}(\vec{r}_2(t)) = <1^2, 1(t), -1(1-t)> = <1, t, t-1>$

$\vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t) = -1 + 0 + t - 1 = t - 2$

$$\int_{C=C_1+C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt + \int_0^1 \vec{F}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt$$

$$= \int_0^1 1\, dt + \int_0^1 (t-2)\, dt$$

$$= t \Big|_{t=0}^{1} + \left(\frac{t^2}{2} - 2t\right)\Big|_{t=0}^{1}$$

$$= 1 - 0 + \frac{1}{2} - 2 - 0 + 0 = \boxed{-\frac{1}{2}}$$

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Problem 3. (4)

Let $\mathbf{F} = (3x^2 - 2xyz^2)\mathbf{i} + (-x^2z^2 + 4y)\mathbf{j} + (-2x^2yz + 3)\mathbf{k}$ defined on $\mathbb{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$ with the orientation given by the parametrization.

(i) $\operatorname{curl} \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$

$= \langle -2x^2z + 2x^2z, -4xyz + 4xyz, -2xz^2 + 2xz^2 \rangle = \vec{0}$

$\rightarrow$ so $\boxed{\vec{F} \text{ is conservative.}}$

(ii) $\frac{\partial V}{\partial x} = F_1 \Rightarrow V = \int F_1\,dx = x^3 - x^2yz^2 + C_1(y,z)$

$\frac{\partial V}{\partial y} = F_2 \Rightarrow V = \int F_2\,dy = -x^2yz^2 + 2y^2 + C_2(x,z)$

$\frac{\partial V}{\partial z} = F_3 \Rightarrow V = \int F_3\,dz = -x^2yz^2 + 3z + C_3(x,y)$

By inspection, $C_1(y,z) = 2y^2 + 3z + C_0$, $C_2(x,z) = x^3 + 3z + C_0$, $C_3(x,y) = x^3 + 2y^2 + C_0$

[illegible] gives us (for some constant C_0)

$\boxed{V(x,y,z) = -x^2yz^2 + x^3 + 2y^2 + 3z + C_0}$

(iii) $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ $\quad \vec{r}(0) = \langle 0, 0, 0 \rangle$ $\quad \vec{r}(1) = \langle 1, 1, 1 \rangle$

Since $\vec{F}$ is conservative,

$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \nabla V(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = V(\vec{r}(1)) - V(\vec{r}(0))$

$= (-1 + 1 + 2 + 3 + C_0) - (0 + 0 + 0 + 0 + C_0) = \boxed{5}$ ✓

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Problem 4. (4)

Evaluate the surface integral $\int\int_S (ax^2+by^2+cxy)dS$ where S is the cone given by the equation $x^2+y^2-z^2=0$ with $0 \le z \le 1$ and a, b and c are positive constants.

[illegible]

$x^2+y^2-z^2=0 \Rightarrow r^2=z^2 \Rightarrow z=r$ (since $r, z \ge 0$)

$G(r,\theta)=\langle r\cos\theta, r\sin\theta, r\rangle \quad D=\{(r,\theta): 0\le r\le 1,\ 0\le\theta\le 2\pi\}$

$\vec{T}_r=\langle\cos\theta, \sin\theta, 1\rangle \quad \vec{T}_\theta=\langle -r\sin\theta, r\cos\theta, 0\rangle$

$$\vec{N}=\vec{T}_r\times\vec{T}_\theta=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix}=\vec{i}(0-r\cos\theta)-\vec{j}(0+r\sin\theta)+\vec{k}(r\cos^2\theta+r\sin^2\theta)$$

$$=\langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$\|\vec{N}\|=\sqrt{r^2\cos^2\theta+r^2\sin^2\theta+r^2}=r\sqrt{2}$$

$$\iint_S (ax^2+by^2+cxy)\,dS=\iint_D \left(a(r\cos\theta)^2+b(r\sin\theta)^2+c(r\cos\theta)(r\sin\theta)\right)\|\vec{N}\|\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^1 r^2\left(a\cos^2\theta+b\sin^2\theta+c\cos\theta\sin\theta\right)(r\sqrt{2})\,dr\,d\theta$$

$$=\int_0^{2\pi}\left(a\cos^2\theta+b\sin^2\theta+c\cos\theta\sin\theta\right)\frac{r^4\sqrt{2}}{4}\Big|_{r=0}^{1}\,d\theta$$

$$=\frac{\sqrt{2}}{4}\int_0^{2\pi}\left(a\cos^2\theta+b\sin^2\theta+\frac{1}{2}c(\sin 2\theta)\right)d\theta$$

Since we are integrating over the full period $0\le\theta\le 2\pi$, by symmetry

$$\int_0^{2\pi}\cos^2\theta\,d\theta=\int_0^{2\pi}\sin^2\theta\,d\theta \quad \text{and} \quad \int_0^{2\pi}\sin(2\theta)\,d\theta=0$$

Therefore,

$$\iint_S (ax^2+by^2+cxy)\,dS=\frac{\sqrt{2}}{4}\int_0^{2\pi}\left(a\left(\frac{\cos^2\theta+\sin^2\theta}{2}\right)+b\left(\frac{\cos^2\theta+\sin^2\theta}{2}\right)+\frac{1}{2}c(0)\right)d\theta$$

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$$=\frac{\sqrt{2}}{4}\int_0^{2\pi}\left(\frac{a}{2}+\frac{b}{2}\right)d\theta=\frac{\sqrt{2}}{8}(a+b)\,\theta\Big|_{\theta=0}^{2\pi}=\boxed{\frac{(a+b)\pi\sqrt{2}}{4}}$$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 - bv^2)$ with domain $D = \{(u,v) \,|\, u^2+v^2 \le 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{n}$ pointing to the negative z-direction (that is the z-component of $\mathbf{n}$ is negative). Find

$$\int\int_S \mathbf{F}\cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = < y, x^2, z >$.

$$\vec{T}_u = \langle 1, 0, 2au \rangle \qquad \vec{T}_v = \langle 0, 1, -2bv \rangle$$

$$\vec{N} = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2au \\ 0 & 1 & -2bv \end{vmatrix} = \vec{i}(-2au) - \vec{j}(-2bv) + \vec{k}(1) = \langle -2au, 2bv, 1 \rangle$$

$N_z > 0$ nope -0.5pt.

$$\vec{F}(G(u,v)) = \langle v, u^2, au^2 - bv^2 \rangle$$

$$\vec{F}\cdot\vec{N} = -2auv + 2bu^2v + au^2 - bv^2$$

Why cartesian?

$$D: \{(u,v) \,|\, u^2+v^2 \le 1\} \to D: \{(u,v) \,|\, -1 \le u \le 1, -\sqrt{1-u^2} \le v \le \sqrt{1-u^2}\}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot\vec{N}\, dv\, du = \int_{-1}^{1}\int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} (-2auv + 2bu^2v + au^2 - bv^2)\, dv\, du$$

$$= \int_{-1}^{1} \left(\cancel{-auv^2} + \cancel{bu^2v^2} + au^2v - \frac{bv^3}{3} \right)\Bigg|_{v=-\sqrt{1-u^2}}^{\sqrt{1-u^2}} du$$

squared terms cancel

$$= \int_{-1}^{1} \left(au^2 - \frac{b(1-u^2)}{3} \right) \sqrt{1-u^2}\, du$$

$u = \cos\theta \quad du = -\sin\theta\, d\theta \quad \sqrt{1-u^2} = \sqrt{1-\cos^2\theta} = \sin\theta$

$$\boxed{= \int_{\pi}^{0} \left(a\cos^2\theta - \frac{b\sin^2\theta}{3} \right)(-\sin\theta\, d\theta)}$$

eh ok.

= do this integral using trig identities

-1pt for no calculation.