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MATH 32B Midterm II, Winter 2019

Name: [redacted]

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

$U=x+y \quad -a\le u\le a \quad Jac(G)=\frac{1}{2}$

$V=x-y \quad -b\le v\le b \quad G(u,v)=(\frac{u+v}{2},\frac{u-v}{2})$

Problem 1. (4)

Evaluate the following integral

$$\int\int_D x^2+2y^2+3x\,dxdy.$$

Here D is the finite region in $\mathbf{R}^2$ bounded by the lines $x+y=a, x+y=-a, x-y=b, x-y=-b$, where a and b are two positive constants.

$u=x+y$, $v=x-y$, $u+v=2x$

$u=x+y$, $-v=-x+y$, $u-v=2y$

$u=x+y \qquad -a\le u\le a \qquad x=\frac{u+v}{2}$

$v=x-y \qquad -b\le v\le b \qquad y=\frac{u-v}{2}$

$G(u,v)=\left(\frac{u+v}{2},\frac{u-v}{2}\right)$

$Jac(G^{-1})=\left|\frac{\partial(u,v)}{\partial(x,y)}\right|=\left|\begin{matrix}1&1\\1&-1\end{matrix}\right|=|-1-1|=2$

$Jac(G)=\frac{1}{2}$

$\iint_D x^2+2y^2+3x\,dxdy=\frac{1}{2}\int_{-b}^{b}\int_{-a}^{a}\left(\frac{u+v}{2}\right)^2+2\left(\frac{u-v}{2}\right)^2+3\left(\frac{u+v}{2}\right)dudv$

$=\frac{1}{8}\int_{-b}^{b}\int_{-a}^{a}u^2+2uv+v^2+2u^2-4uv+2v^2+6u+6v\,dudv$

$=\frac{1}{8}\int_{-b}^{b}\int_{-a}^{a}3u^2+3v^2-2uv+6u+6v\,dudv$

$=\frac{1}{8}\int_{-b}^{b}\left(u^3+3uv^2-u^2v+3u^2+6uv\right)_{-a}^{a}dv$

$(a^3+3av^2-\cancel{a^2v}+\cancel{3a^2}+6av)-(-a^3-3av^2-\cancel{a^2v}+\cancel{3a^2}-6av)$

$=\frac{1}{4}\int_{-b}^{b}a^3+3av^2+6av\,dv=\frac{1}{4}\left(a^3v+av^3+3av^2\right)_{-b}^{b}$

$=\frac{1}{4}\left[(ba^3+ab^3+\cancel{3ab^2})-(-a^3b-ab^3+\cancel{3ab^2})\right]$

$=\boxed{\frac{1}{2}(a^3b+ab^3)}$

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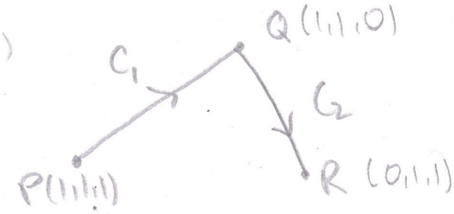
$C_2: \vec{r}(t) = (1-t)(1,1,0) + t(0,1,1)$
$= \langle 1-t, 1-t, 0\rangle + \langle 0, t, t\rangle$
$= \langle 1-t, 1, t\rangle$

$\vec{F}(\vec{r}(t)) = \langle 1, t, t-1\rangle$



$C_1: \vec{r}(t) = (1-t)(1,1,1) + t(1,1,0)$
$= \langle 1-t, 1-t, 1-t\rangle + \langle t, t, 0\rangle = \langle 1, 1, 1-t\rangle$

Problem 2. (4)

Let $\mathbf{F} =< y^2, yz, -xy >$. Find the line integral $\int_{C=C_1+C_2} \mathbf{F} \cdot d\mathbf{r}$, where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1, 1, 1)$, $Q = (1, 1, 0)$ and $R = (0, 1, 1)$.

$$\int_{C=C_1+C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$C_1: \quad \vec{r}(t) = (1-t)(1,1,1) + t(1,1,0)$
$= (1,1,1-t)$

$\vec{F}(\vec{r}(t)) = \langle 1, 1-t, -1\rangle$

$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 1, 1-t, -1\rangle \cdot \langle 0, 0, -$
$= 1$

$\vec{r}\,'(t) = (0, 0, -1) \quad 0 \le t \le 1$

$\int_0^1 1\, dt = 1$

$C_2: \quad \vec{r}_2(t) = (1-t)(1,1,0) + t(0,1,1)$
$= \langle 1-t, 1, t\rangle \quad 0 \le t \le 1$

$\vec{r}_2\,'(t) = \langle -1, 0, 1\rangle$

$\vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t) = \langle 1, t, t-1\rangle \cdot \langle -1, 0, 1\rangle$
$= -1 + t - 1 = t - 2$

$$\int_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 t - 2\, dt = \left(\frac{t^2}{2} - 2t\right)\Big|_0^1 = \left(\frac{1}{2} - 2\right) = -\frac{3}{2}$$

$$\int_{C=C_1+C_2} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r} = 1 + \frac{-3}{2} = \boxed{-\frac{1}{2}}$$

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Problem 3. (4)

Let $\mathbf{F} = (3x^2 - 2xyz^2)\mathbf{i} + (-x^2z^2 + 4y)\mathbf{j} + (-2x^2yz + 3)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot dr$, where C is given by the parametric equation $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$ with the orientation given by the parametrization.

$\partial_x V = 3x^2 - 2xyz^2$
$\partial_y V = -x^2z^2 + 4y$
$\partial_z V = -2x^2yz + 3$

$\vec{F} = \langle 3x^2 - 2xyz^2, -x^2z^2 + 4y, -2x^2yz + 3 \rangle$

$V = \int F_1\, dx = \int 3x^2 - 2xyz^2\, dx = x^3 - x^2yz^2 + C(y,z)$

$\frac{\partial V}{\partial y} = \cancel{-x^2z^2} + \partial_y C = \cancel{-x^2z^2} + 4y$

$\partial_y C = 4y \qquad C = \int 4y\,dy = 2y^2 + C(z).$

$V = x^3 - x^2yz^2 + 2y^2 + C(z)$

$\frac{\partial V}{\partial z} = \cancel{-2x^2yz} + \frac{dC}{dz} = \cancel{-2x^2yz} + 3 \qquad C = \int 3\,dz = 3z.$

$V = x^3 - x^2yz^2 + 2y^2 + 3z$

i) $\vec{F}$ is conservative b/c there is a potential V s.t. $\nabla V = \vec{F}$

ii) $\boxed{V = x^3 - x^2yz^2 + 2y^2 + 3z}$

iii) $\int_C \vec{F} \cdot d\vec{r}$ $\quad C$ is parameterized by $\vec{r}(t) = (t, t^2, t^3)\quad 0 \le t \le 1$

$\vec{r}(0) = (0,0,0) \quad \vec{r}(1) = (1,1,1)$

Since $\vec{F}$ is conservative, $\int_C \vec{F} \cdot d\vec{r} = V(1,1,1) - V(0,0,0)$

$= (1 - 1 + 2 + 3) - (0)$

$= \boxed{5}$ ✓

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Problem 4. (4)

Evaluate the surface integral $\int\int_S \overbrace{(ax^2+by^2+cxy)}^{f}dS$ where S is the cone given by the equation $x^2+y^2-z^2=0$ with $0 \le z \le 1$ and a, b and c are positive constants.

$x^2+y^2=z^2$

$r^2 = z^2 \Rightarrow r=z$

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

$D: \{(r,\theta) \mid 0 \le r \le 1, 0 \le \theta \le 2\pi$

$\vec{N} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -\sin\theta & \cos\theta & 0 \end{vmatrix}$ r

$= r\langle -\cos\theta, -\sin\theta, 1 \rangle$

$\|\vec{N}\| = \sqrt{2}\, r$

$f = ax^2+by^2+cxy$

$f(G(r,\theta)) = ar^2\cos^2\theta + br^2\sin^2\theta + cr^2\cos\theta\sin\theta$

$\iint_S f\, dS = \sqrt{2}\int_0^{2\pi}\int_0^1 r^3(a\cos^2\theta + b\sin^2\theta + c\cos\theta\sin\theta)\, dr\, d\theta$

$= \sqrt{2} \int_0^{2\pi} a\cos^2\theta + b\sin^2\theta + \cancel{c\cos\theta\sin\theta}\, d\theta \int_0^1 r^3\, dr$

$=0$ b/c full period of $\cos\theta$ & $\sin\theta = 0$

$= \sqrt{2}\,(a\pi + b\pi)\left(\frac{1}{4}\right)$

$= \boxed{\frac{\sqrt{2}}{4}(a\pi + b\pi)}$ ✓

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, r \rangle$

$\|\vec{N}\| = \sqrt{2}\, r$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u, v) = (u, v, au^2 - bv^2)$ with domain $D = \{(u, v) | u^2 + v^2 \leq 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{n}$ pointing to the negative z-direction (that is the z-component of $\mathbf{n}$ is negative). Find

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} =< y, x^2, z >$. $= F(G(u,v)) = \langle v, u^2, au^2 - bv^2 \rangle$

$\partial_u G = \langle 1, 0, 2au \rangle$

$\partial_v G = \langle 0, 1, -2bv \rangle$

$\vec{N} = \begin{vmatrix} i & j & k \\ 1 & 0 & 2au \\ 0 & 1 & -2bv \end{vmatrix}$

$= \langle -2au, 2bv, 1 \rangle$

$= \langle 2au, -2bv, -1 \rangle$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \langle v, u^2, au^2 - bv^2 \rangle \cdot \langle 2au, -2bv, -1 \rangle \, du\,dv$

$= \iint_D \underbrace{2auv - 2bvu^2 - au^2 + bv^2}_{g(u,v)} \, du\,dv$

$D : \{(u,v) \mid u^2 + v^2 \leq 1\}$
$r^2 \leq 1$

$\downarrow$

$D_0 = \{(r, \theta) \mid 0 \leq r \leq 1 \quad 0 \leq \theta \leq 2\pi\}$ $\quad H(r,\theta) = (r\cos\theta, r\sin\theta)$

$\mathrm{Jac}(H) = r$

$g(H(r,\theta)) = 2a(r\cos\theta)(r\sin\theta) - 2b(r\cos\theta)^2(r\sin\theta)$
$- a(r\cos\theta)^2 + b(r\sin\theta)^2$

$= 2ar^2\cos\theta\sin\theta - 2br^3\cos^2\theta\sin\theta - ar^2\cos^2\theta + br^2\sin^2\theta$

$\iint_{D_0} g(H(r,\theta))\, r \, dr\,d\theta$

$= \int_0^1\int_0^{2\pi} \cancel{2ar^3\cos\theta\sin\theta} - \cancel{2br^4\cos^2\theta\sin\theta} - ar^3\cos^2\theta + br^3\sin^2\theta \, d\theta\,dr$

$\cancel{\int_0^{2\pi}\int_0^1 -2br^4\cos^2\theta\sin\theta\,dr\,d\theta}$

$\cancel{u = \cos\theta}$

$\cancel{du = -\sin\theta\,d\theta}$

$\cancel{\int_0^1 r^4\,dr = \frac{1}{5}}$

$= \int_0^1 -ar^3\pi + br^3\pi \, dr$

$= -a\pi + b\pi \cdot \int_0^1 r^3\,dr$

$= \frac{1}{4}(b\pi - a\pi)$

$\boxed{\frac{\pi}{4}(b-a)}$