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MATH 32B Midterm II, Winter 2019

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Evaluate the following integral

$$I = \int\int_{\mathbf{D}} x^2 + 2y^2 + 3x\, dxdy.$$

let $u = x+y$, $v = x-y$.

Here D is the finite region in $\mathbf{R}^2$ bounded by the lines $x+y=a, x+y=-a, x-y=b, x-y=-b$, where a and b are two positive constants.

$-a \le u \le a$
$-b \le v \le b$

$x = \frac{u+v}{2}$, $y = \frac{u-v}{2}$

$\text{Jac}(G) = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left| -\frac{1}{4} - \frac{1}{4} \right| = \frac{1}{2}$

$I = \frac{1}{2} \int_{-a}^{a}\int_{-b}^{b} \left(\frac{1}{4}(u+v)^2 + \frac{1}{2}(u-v)^2 + \frac{3}{2}(u+v)\right) dv\,du$

$= \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b} \left(\frac{1}{4}(u^2+2uv+v^2) + \frac{1}{2}(u^2-2uv+v^2) + \frac{3}{2}(u+v)\right) dv\,du$

$= \frac{1}{2}\int_{-a}^{a}\int_{-b}^{b} \left(\frac{3}{4}u^2 + \frac{3}{4}v^2 - \frac{1}{2}uv + \frac{3}{2}u + \frac{3}{2}v\right) dv\,du$

$= \frac{1}{2}\left[2b \cdot \frac{1}{4}u^3\Big|_{-a}^{a} + 2a\frac{1}{4}v^3\Big|_{-b}^{b} + \underbrace{2b\frac{3}{4}u^2\Big|_{-a}^{a} + 2b\frac{3}{4}v^2\Big|_{-b}^{b}}_{0} \right. + \frac{1}{2}\underbrace{\left(\frac{1}{2}u^2\right)\Big|_{-a}^{a}\left(\frac{1}{2}v^2\right)\Big|_{-b}^{b}}_{0}\Big]$

$= \boxed{\frac{1}{2}ba^3 + \frac{1}{2}ab^3}$

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Problem 2. (4)

Let $\mathbf{F} =< y^2, yz, -xy >$. Find the line integral $\int_{C=C_1+C_2} \mathbf{F}\cdot d\mathbf{r}$, where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,1,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

$P \to Q \to R$

$r_1(t) = (1, 1, 1-t) \quad 0 \le t \le 1.$

$r_1'(t) = \langle 0, 0, -1 \rangle. \quad F(r_1(t)) = \langle 1, 1-t, -1 \rangle$

$\int_{C_1} F \cdot dr = \int_0^1 1 \, dt = t \Big|_0^1 = 1$

$r_2(t) = \langle 1-t, 1, t \rangle \quad 0 \le t \le 1.$

$r_2'(t) = \langle -1, 0, 1 \rangle. \quad F(r_2(t)) = \langle 1, t, t-1 \rangle$

$\int_{C_2} F \cdot dr = \int_0^1 (-1 + t - 1)\, dt = \int_0^1 (t-2)\, dt$

$= \left(\frac{1}{2}t^2 - 2t\right)\Big|_0^1$

$= \frac{1}{2} - 2 = -\frac{3}{2}.$

$\int_{C_1+C_2} F \cdot dr = 1 + \left(-\frac{3}{2}\right) = -\frac{1}{2}.$

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Problem 3. (4)

Let $\mathbf{F} = (3x^2 - 2xyz^2)\mathbf{i} + (-x^2z^2 + 4y)\mathbf{j} + (-2x^2yz + 3)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot dr$, where C is given by the parametric equation $x = t,\ y = t^2,\ z = t^3,\ 0 \le t \le 1$ with the orientation given by the parametrization.

(i) $\partial_y F_x = -2xz^2 = \partial_x F_y$ ✓

$\partial_x F_z = -4xyz = \partial_z F_x$. $\Rightarrow$ F is conservative

$\partial_y F_z = -2x^2 z = \partial_z F_y$ ✓

(ii) By inspection, $V = x^3 - x^2 y z^2 + 2y^2 + 3z$.

where $F = \nabla V$.

(iii) Since F is conservative,

$\int_C F \cdot dr = V(r(1)) - V(r(0))$

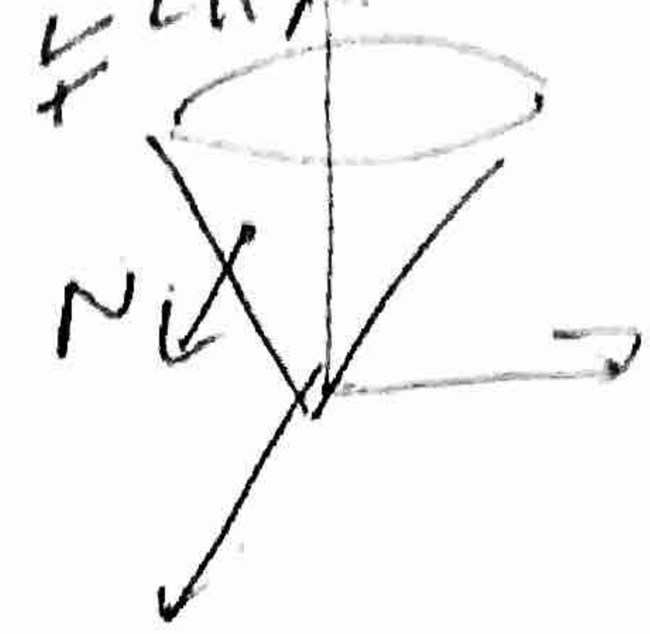
$= V(1,1,1) - V(0,0,0)$

$= (1 - 1 + 2 + 3) = \boxed{5}$

Cone: $z^2 = x^2 + y^2$. $\quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$G(r,\theta) = (r\cos\theta, r\sin\theta, r)$

$N = (r\cos\theta, r\sin\theta, -r)$



$\|N\| = \sqrt{2}\, r$

4 **Problem 4.** (4)

F:

Evaluate the surface integral $\int\int_S (ax^2 + by^2 + cxy)dS$ where S is the cone given by the equation $x^2 + y^2 - z^2 = 0$ with $0 \le z \le 1$ and a, b and c are positive constants.

$$\int\int_S (ax^2 + by^2 + cxy)\,dS$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 (a r^2\cos^2\theta + b r^2\sin^2\theta + c r^2\cos\theta\sin\theta)\, r\, dr\, d\theta$$

$$= \sqrt{2}\left[a\int_0^1 r^3\,dr \int_0^{2\pi}\cos^2\theta\, d\theta + b\int_0^1 r^3\,dr\int_0^{2\pi}\sin^2\theta\, d\theta + c\int_0^1 r^2\,dr\int_0^{2\pi}\cos\theta\sin\theta\, d\theta\right.$$

$$= \sqrt{2}\left(a\left(\tfrac{1}{4}\right)(\pi) + b\left(\tfrac{1}{4}\right)(\pi) + \underbrace{c\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{2}\cos^2\theta\right)\Big|_0^{2\pi}}_{0}\right)$$

$$= \sqrt{2}\,\pi\left(\frac{a}{4} + \frac{b}{4}\right)$$

let: $u = r\cos\theta$ $\quad 0 \le r \le 1$
$v = r\sin\theta$ $\quad 0 \le \theta \le 2\pi$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 - bv^2)$ with domain $D = \{(u,v) \mid u^2 + v^2 \le 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{n}$ pointing to the negative z-direction (that is the z-component of $\mathbf{n}$ is negative). Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = \langle y, x^2, z \rangle$.

$T_u = (1, 0, 2au)$, $T_v = (0, 1, -2bv)$

negative

$N = T_v \times T_u = \begin{vmatrix} i & j & k \\ 0 & 1 & -2bv \\ 1 & 0 & 2au \end{vmatrix} = \langle 2au, -2bv, -1 \rangle$

$F(G(u,v)) = \langle v, u^2, au^2 - bv^2 \rangle$

$$\iint_S F \cdot dS = \iint_S (\underbrace{2auv - 2bu^2v}_{0} - au^2 + bv^2)\, du\, dv$$

$$= \int_0^{2\pi}\int_0^1 (2a\, r^2\cos\theta\sin\theta - 2b\, r^3\cos^2\theta\sin\theta - a r^2\cos^2\theta + b r^2\sin^2\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 -a r^3\cos^2\theta + b r^3\sin^2\theta \, dr\, d\theta$$

$$= -a\left(\tfrac{1}{4}\right)\pi + b\left(\tfrac{1}{4}\right)\pi$$