1	2.5
2	4
3	4
4	2.5
5	3
T	16

32B Midterm II, Winter 2017

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...wers. No Points Will Be Given Without Suffi-
...ng/Calculations.

Problem 1. (4)

Evaluate the following integral

$$\int\int_{D} x^2 + 2y^2 dxdy.$$

$u=a \Updownarrow$ $\quad u=-a \Updownarrow$

Here D is the finite region in $\mathbf{R}^2$ bounded by the lines $x+y=a, x+y=-a, x-y=b, x-y=-b$, where a and b are two positive constants.

$x-y=b \Updownarrow v=b$ $\quad x-y=-b \Updownarrow v=-b$

Let $u = x+y,\ v = x-y$

$u+v = 2x \Rightarrow x = \frac{1}{2}(u+v)$

$u-v = 2y \Rightarrow y = \frac{1}{2}(u-v)$ $\quad +1$

$D = \{(u,v);\ -a \le u \le a,\ -b \le v \le b\}$

$\mathrm{Jac}(G) = \left|\det\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix}\right| = \left|-\frac{1}{2}-\frac{1}{2}\right| = |-1| = 1$

$\frac{1}{4}(u^2+2uv+v^2) + \frac{1}{2}(u^2-2uv+v^2)$

$\iint_D = x^2+2y^2\,dxdy = \int_{-a}^{a}\int_{-b}^{b}\left(\left(\frac{1}{2}\right)^2(u+v)^2 + 2\left(\frac{1}{2}\right)^2(u-v)^2\right)du\,dv$

$= \int_{-a}^{a}\int_{-b}^{b}\left(\frac{3}{4}u^2 - \frac{1}{2}uv + \frac{3}{4}v^2\right)du\,dv$

$= \int_{-a}^{a}\left(\frac{1}{4}u^3 - \frac{1}{4}u^2v + \frac{3}{4}v^2u\right)\Big|_{-b}^{b} dv$

$= \int_{-a}^{a}\left(\frac{b^3}{4} - \cancel{\frac{1}{4}b^2v} + \frac{3}{4}v^2b - \left(-\frac{b^3}{4} - \cancel{\frac{1}{4}b^2v} + \frac{3b}{4}v^2\right)\right)dv$

1 $= \int_{-a}^{a}\left(\frac{1}{2}b^3 + \frac{3}{2}bv^2\right)dv = \frac{1}{2}b^3v + \frac{1}{2}bv^3\Big|_{-a}^{a}$

$= b^3a + ba^3$ $\quad +1.5$

Problem 2. (4)

Find the line integral

$$\int_{C_1+C_2} yzdx + xzdy + xydz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,0,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

$\frac{\partial F_1}{\partial y} = z = \frac{\partial F_2}{\partial x}$ ✓ $\quad F = \langle yz, xz, xy \rangle$ is conservative

$\frac{\partial F_1}{\partial z} = y = \frac{\partial F_3}{\partial x}$ ✓ $\quad \int F_1 dx = xyz + C_1(y,z)$ $\quad$ By introspection,

$\int F_2 dy = xyz + C_2(x,z)$ $\quad f(x,y,z) = xyz$

$\frac{\partial F_2}{\partial z} = x = \frac{\partial F_3}{\partial y}$ ✓ $\quad \int F_3 dz = xyz + C_3(x,z)$

$\int_{C_1+C_2} yzdx + xzdy + xydz = f(R) - f(P) = 0 - 0 = 0$ ✓

If conserv.

$\hookrightarrow \int_P^Q F dr + \int_Q^R F dr = f(Q) - f(P) + f(R) - f(Q) = f(R) - f(P)$

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Problem 3. (4)

Let $\mathbf{F} = (x^3 + y)\mathbf{i} + (x + z - \sin y)\mathbf{j} + (z^2 + y + \cos z)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t^2$, $y = t^4$, $z = t^6$, $0 \leq t \leq 1$ with the orientation given by the parametrization.

I) $\frac{\partial F_1}{\partial y} = 1$ $\quad \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ ✓ $\quad \frac{\partial F_2}{\partial z} = 1$ $\quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$ ✓ $\quad \frac{\partial F_1}{\partial z} = 0$ $\quad \frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ ✓

$\frac{\partial F_2}{\partial x} = 1$ $\quad \frac{\partial F_3}{\partial y} = 1$ $\quad \frac{\partial F_3}{\partial x} = 0$

✓

F is conservative

II) $\int F_1 dx = \frac{1}{4}x^4 + xy + c_1(\cancel{y},z)$ $\qquad \frac{\partial(\int F_1 dx)}{\partial y} = x + c_1'(y,z) = x + z - \sin y$

$c_1'(y,z) = \cancel{x} + z - \sin y$

$c_1(y,z) = \cancel{xy} + zy + \cos y$

✓

$\frac{\partial f}{\partial z} = \left(\frac{1}{4}x^4 + xy + zy + \cos y\right)\frac{\partial}{\partial z}$

$y + c_1'(z) = z^2 + y + \cos z$

$c_1'(z) = z^2 + \cos z$

$c_1(z) = \frac{1}{3}z^3 + \sin z$

$f(x,y,z) = \frac{1}{4}x^4 + xy + zy + \cos y + \frac{1}{3}z^3 + \sin z$

III) $r(t) = (t^2, t^4, t^6)$ $\quad r(1) = (1,1,1)$ $\quad r(0) = (0,0,0)$

$\int_C F \cdot dr = f(r(1)) - f(r(0)) = \left(\frac{1}{4} + 1 + 1 + \cos 1 + \frac{1}{3} + \sin 1\right) - (1) = \frac{19}{12} + \cos 1 + \sin 1$

✓

Problem 4. (4)

Evaluate the surface integral $\int\int_S (3x^2 + 4y^2 + 5z^2)dS$ where S is the sphere given by the equation $x^2 + y^2 + z^2 = a^2$ where a is a positive constant.

$\iint_S (3x^2+4y^2+5z^2)ds = \iint_S (3a^2+y^2+2z^2)dS$ $\quad G(\theta,\phi) = a\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle$

$\|\hat{N}\| = a^2\sin\phi$

$\iint_S (F(G(\theta,\phi))\|\hat{N}\| ds$

(0, 2π) & region is sym. ~~odd~~ in θ across θ so it's 0

$\|\hat{N}\|$

$\iint_S (3a^2 + a^2\sin^2\phi\cos^2\theta + a^2 2\cos^2\phi)ds = a^4 \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} (3+\sin^2\phi\cos^2\theta + 2\cos^2\phi)\sin\phi \, d\phi \, d\theta$

$\cos^2\theta$

$= a^4 \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\frac{\pi}{2}} (\sin^3\phi\cos^2\theta)\,d\phi\,d\theta = a^4 \int_0^{2\pi}\int_0^{\frac{\pi}{2}} (1-\cos^2\phi)\sin\phi\,d\phi\,d\theta$

$= a^4 \left(\frac{\theta}{2} + \frac{\sin\theta\cos\theta}{2}\right)\Big|_0^{2\pi} \left(-\cos\phi + \frac{\cos^3\phi}{3}\right)\Big|_0^{\frac{\pi}{2}} = a^4(\pi+0-0)((0+0)-(-1+\frac{1}{3})) = \cancel{\frac{2a^4\pi}{3}}$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 + bv^2)$ with domain $D = \{(u,v) \,|\, u^2 + v^2 \leq 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{N}$ pointing to the negative z-direction (that is the z-component of $\mathbf{N}$ is negative). Find

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

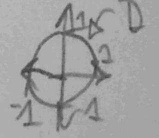
Here the vector field $\mathbf{F} = < z, x^2, x^2 >$. $G_u = \langle 1, 0, 2au \rangle$ $+N = (G_u \times G_v) = -1 \langle -2au, 2bv, 1 \rangle$

$F(G(u,v)) = \langle au^2 + bv^2, u^2, u^2 \rangle$ $G_v = \langle 0, 1, 2bv \rangle$ $N = \langle 2au, -2bv, -1 \rangle$

is not odd

$\iint_S F \, dS = \iint_D F(G(u,v)) \cdot N \, dA = \iint_D \langle 2a^2u^3 + 2abv^2u, -2bvu^2, -u^2 \rangle \, dA$

Region is symmetric in u & v & each component of $F(G(u,v)) \cdot N$ is odd in either u or v, therefore



$\iint_S F \cdot dS = 0$ X -1