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2	3.5
3	4
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MATH 32B Midterm II, Spring 2015

Name:

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the line integral

$$\int_{C_1+C_2} x^2dx + ydy + z^2dz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (0,0,1)$, $Q = (1,1,2)$ and $R = (1,2,3)$.

C_1:

$x(t) = (1-t)0 + t = t$ $\quad x'(t) = 1\,dt$

$y(t) = t$ $\quad y'(t) = 1\,dt$

$z(t) = (1-t)\cdot 1 + 2t = 1-t+2t = t+1$ $\quad z'(t) = 1\,dt$

C_2: $x(t) = (1-t)1 + 1t = 1$ $\quad x'(t) = 0\,dt$

$y(t) = (1-t)1 + 2t = t+1$ $\quad y'(t) = 1\,dt$

$z(t) = (1-t)2 + 3t = t+2$ $\quad z'(t) = 1\,dt$

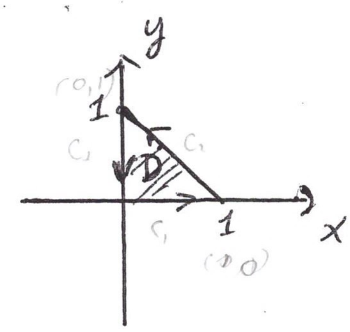
$\int_0^1 t^2dt + t\,dt + (t+1)^2dt + \int_0^1 0 + t+1\,dt + (t+2)^2dt$

$\int_0^1 (t^2 + t + t^2 + 2t + 1)dt + \int_0^1 t+1 + t^2 + 4t + 4\,dt =$

$= \int_0^1 2t^2 + 3t + 1\,dt + \int_0^1 t^2 + 5t + 5\,dt$

$= \frac{2t^3}{3} + \frac{3t^2}{2} + t\Big|_0^1 + \frac{t^3}{3} + \frac{5t^2}{2} + 5t\Big|_0^1$

$= \frac{2}{3} + \frac{3}{2} + 1 + \frac{1}{3} + \frac{5}{2} + 5 = 4 + 1 + 6 = \boxed{11}$



$D: x: 0 \le x \le 1 \qquad \int F_1\,dx + F_2\,dy$

$y: 1-x \le y \le 1 \qquad = \iint F_{2x} - F_{1y}$

Problem 2. (4)

Evaluate the following line integral

$$\int_{\mathbf{C}} 2ydx - xdy$$

by two different methods:(a) compute it directly; (b) use the Green theorem to compute it. Here **C** is the boundary of the triangle D with the counter clockwise orientation.

C_1: $x(t) = t \Rightarrow 1$, $y(t) = 0 \Rightarrow 0$, $\langle t, 0\rangle$

C_2: $x(t) = (1-t)1 + 0 = 1-t \Rightarrow -1\,dt$, $y(t) = (1-t)0 + t = t \Rightarrow 1$, $\langle 1-t, t\rangle$ $\langle -1, 1\rangle$

C_3: $x(t) = 0 \Rightarrow 0$, $y(t) = (1-t)1 + 0 = 1-t \Rightarrow -1$, $\langle 0, 1-t\rangle$ $\langle 0, -1\rangle$

$\int_0^1 2\cdot 0 \cdot 1 - t\cdot 0\,dt + \int_0^1 2t(-1)\,dt - (1-t)\,dt + \int_0^1 2(1-t)\cdot 0 - 0$

$= \int_0^1 -2t - 1 + t\,dt = \int_0^1 -t - 1\,dt$

$-\frac{t^2}{2} - t \Big|_0^1$

$-\frac{1}{2} - 1 = -\frac{3}{2}$

$F_1 \quad \langle 2y, -x\rangle$

$\int_C 2y\,dx - x\,dy$

$\int_0^1 \int_{y=1-x}^{y=1} (-1+2)\,dy\,dx$ X

$= \int_0^1 y \Big|_{1-x}^{1} dx$

$= \int_0^1 1 - (1-x)\,dx = \int_0^1 x\,dx$

$= \frac{x^2}{2}\Big|_0^1 = \boxed{\frac{1}{2}}$ $\quad -0.5$

$F_1 \quad F_2 \quad F_3$

$\langle 2xy^2z^2+y+3z,\ 2x^2yz^2+x+z,\ 2x^2y^2z+3x+y \rangle$

Problem 3. (4)

Let $F = (2xy^2z^2 + y + 3z)\mathbf{i} + (2x^2yz^2 + x + z)\mathbf{j} + (2x^2y^2z + 3x + y)\mathbf{k}$ defined on $\mathbf{R}^3$.

(i) Decide whether or not F is conservative.

(ii) If F is conservative, find the potential function V, such that $F = \nabla V$.

(iii) Compute the line integral $\int_C F \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t,\ y = t^2,\ z = t^3\ 0 \leq t \leq 1$ with the orientation given by the parametrization.

$\frac{dF_1}{dy} = 4xyz^2+1 \qquad \frac{dF_2}{dx} = 4xyz^2+1, \qquad \frac{dF_1}{dz} = 4xy^2z+3 \qquad \frac{dF_3}{dx} = 4xy^2z+3$ ✓

$\frac{dF_2}{dz} = 4x^2yz+1 \qquad \frac{dF_3}{dy} = 4x^2yz+1$ ✓ it is conservative

$\int 2xy^2z^2+y+3z\,dx = x^2y^2z^2+xy+3zx+C(y,z)$

$F_2 = 2x^2yz^2+x+z = 2x^2yz^2+x+C'(y,z)$

$C(y,z) = yz + C(z)$

$2x^2y^2z+3x+y = 2x^2y^2z+3x+y+C'(z)$

$V = x^2y^2z^2+xy+3zx+yz$

$\int_C F\cdot dr = \int_C \nabla V \cdot dr = V(1,1,1) - V(0,0,0) = \boxed{6}$

$V(1,1,1) = 1+1+3+1 = 6$

$V(0,0,0) = 0+0+0+0 = 0$

Problem 4. (4)

Evaluate the surface integral $\int\int_S (x^2 + y^2) dS$ where S is the sphere given by the equation $x^2 + y^2 + z^2 = a^2$.

$G(\varphi, \theta) = (a\cos\theta\sin\varphi,\ a\sin\theta\sin\varphi,\ a\cos\varphi)$

$0 \le \varphi \le \pi$

$0 \le \theta \le 2\pi$

$\|n\| = a^2 \sin\varphi$

$x^2 + y^2 = a^2\cos^2\theta\sin^2\varphi + a^2\sin^2\theta\sin^2\varphi = a^2\sin^2\varphi$

$$\int_0^{2\pi}\int_0^{\pi} a^2\sin^2\varphi\,(a^2\sin\varphi)\,d\varphi\,d\theta$$

$(1-\cos^2\varphi)\cdot\sin\varphi$

$$= 2\pi a^4 \cdot \int_0^{\pi} \sin^3\varphi\,d\varphi = 2\pi a^4 \int_0^{\pi} \sin\varphi - \cos^2\varphi\sin\varphi$$

$$= 2\pi a^4 \left(-\cos\varphi + \frac{\cos^3\varphi}{3}\right|_0^{\pi}$$

$$= 2\pi a^4 \left(-(-1) + \frac{-1}{3} - \left(-1 + \frac{1}{3}\right)\right)$$

$$= 2\pi a^4 \left(1 - \frac{1}{3} - \left(-\frac{2}{3}\right)\right)$$

$$= 2\pi a^4 \left(\frac{4}{3}\right)$$

$$= \boxed{\frac{8\pi a^4}{3}}$$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u, v) = (u, v, u^2 - 2v^2)$ with domain $D = \{(u, v) |\, u^2 + v^2 \leq 1\}$. Orient S with the normal pointing to the negative z-direction. Find the flux

$u = r\cos\theta$ $\quad$ $0 \le \theta \le 2\pi$

$v = r\sin\theta$ $\quad$ $0 \le r \le 1$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} =< z, x, y^2 >$.

$T_u = (1, 0, 2u)$

$T_v = (0, 1, -4v)$

$$n = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & 2u \\ 0 & 1 & -4v \end{vmatrix} = (-2u)\hat{i} - \hat{j}(-4v) + \hat{k}(1)$$

$n = <-2u, 4v, 1>$

$F \cdot n = <u^2 - 2v^2, u, v^2> \cdot <-2u, 4v, 1>$

$= -2u^3 + 4v^2u + 4vu + v^2$

$= -2r^3\cos^3\theta + 4r^3\sin^2\theta\cos\theta + 4r^2\sin\theta\cos\theta + r^2\cos^2\theta$

$\int\cos^3\theta$

$\int(1-\sin^2\theta)\cos\theta$

$\int\cos\theta - \sin^2\theta\cos\theta$

$\sin\theta - \frac{\sin^3\theta}{3}$

$\int\cos^2\theta$

$\frac{1+\cos 2\theta}{2}$

$$\int_0^1\int_0^{2\pi}\left(-2r^3\cos^3\theta + 4r^3\sin^2\theta\cos\theta + 4r^2\sin\theta\cos\theta + r^2\underbrace{\cos^2\theta}_{\pi}\right) r\,d\theta\,dr$$

$$= \int_0^1 -2r^4\left(\sin\theta - \frac{\sin^3\theta}{3}\right)\Big|_0^{2\pi} + 4r^4\left(\frac{\sin^3\theta}{3}\right)\Big|_0^{2\pi} + 4r^3\left(\frac{\sin^2\theta}{2}\right)\Big|_0^{2\pi} + r^3 \cdot \pi \, dr$$

$$= \int_0^1 -2r^4(0) + 4r^4(0) + 4r^3(0) + r^3\pi \, dr$$

$$= \pi\frac{r^4}{4}\Big|_0^1 = \boxed{\frac{\pi}{4}}$$