1	4
2	3
3	4
4	4
5	4
T	19

$$a\left(\frac{u^3 v}{3}+\frac{uv^3}{3}\right)-b\left(\frac{u^3v}{3}+\frac{uv^3}{3}\right)$$

$$\frac{1}{2}\left(\frac{a^4b}{3}+\frac{a^2b^3}{3}-\frac{a^3b^2}{3}+\frac{ab^4}{3}\right)$$

MATH 32B Midterm II, Fall 2018

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Evaluate the following integral

$$\iint_D ax^2 - by^2 dxdy. \quad = abx^2 - aby^2$$

$a^2 = ax + ay$
$b^2 = by - by$

Here D is the finite region in $\mathbf{R}^2$ bounded by the lines $x+y=a, x+y=-a, x-y=b, x-y=-b$, where a and b are two positive constants.

$u = x+y \quad \mathrm{Jac}(G) = \frac{1}{2}$

$v = x-y$

$x = \frac{u+v}{2} \quad \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a} a\left(\frac{u+v}{2}\right)^2 - b\left(\frac{u-v}{2}\right)^2 du\,dv =$

$y = \frac{u-v}{2}$

4

$$\frac{1}{2}\int_{-b}^{b}\int_{-a}^{a} \frac{a}{4}(u^2+2uv+v^2) - \frac{b}{4}(u^2-2uv+v^2)\,du\,dv =$$

$$\frac{1}{8}\int_{-b}^{b} a\left(\frac{u^3}{3}+\cancel{2u^2v}+uv^2\right) - b\left(\frac{u^3}{3}-\cancel{2uv}+uv^2\right) dv \Big|_{u=-a}^{a}$$

$$\frac{1}{4}\int_{-b}^{b} \frac{a^4}{3} + a^2v^2 - b\left(\frac{a^3}{3}+av^2\right) dv =$$

$$\frac{1}{4}\left[\left(\frac{a^4 v}{3}+\frac{a^2v^3}{3}\right) - b\left(\frac{a^3v}{3}+\frac{av^3}{3}\right)\right]\Big|_{v=-b}^{b} =$$

$$\frac{1}{2}\left[\frac{a^4b}{3}+\frac{a^2b^3}{3}-\frac{a^3b^2}{3}-\frac{ab^4}{3}\right] =$$

$$\frac{1}{6}\left(a^4b + a^2b^3 - a^3b^2 - ab^4\right)$$

3

Problem 2. (4) Find the line integral

$$\int_{C_1+C_2} xy\,dx - xz\,dy + 2yz\,dz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,1,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

$0 \le t \le 1$

$r_1(t) = (1,1,1-t)$ $r_1'(t) = \langle 0,0,-1\rangle \cdot F(r_1(t)) = (1, t-1, 2-2t) = \boxed{2t-2}$

• dot prod

$r_2(t) = (1-t, 1, t)$ $r_2'(t) = \langle -1,0,1\rangle$ $F(r_2(t)) = (1-t, t^2-t, \boxed{0}) = t-1$

$2t$

(-1)

$\int_0^t t-1\,dt = \frac{t^2}{2} - t \Big|_0^1 = -\frac{1}{2}$

$+$

$\int_0^1 2t-2\,dt = t^2 - 2t\Big|_0^1 = -1$

$= -3/2$

Problem 3. (4)

Let $\mathbf{F} = (3x^2 - 4xyz^2)\mathbf{i} + (-2x^2z^2 + 2y)\mathbf{j} + (-4x^2yz + 1)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$ with the orientation given by the parametrization.

(i) $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = -4xz^2$ ✓ $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = -4x^2z$ ✓ $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} = -8xyz$ ✓

curl$(F) = 0$
F is simply connected on domain D.

$\therefore$ F is conservative

not relevant

(ii) $x^3 - 2x^2yz^2 + y^2 + z + \cancel{C} = V$ by introspection

(iii) $r(0) = (0,0,0)$ $\quad V(1) - V(0) = (1-2+1+1) - (0-0+0+0)$
$r(1) = (1,1,1)$ $\quad = 1$

✓

4

Problem 4. (4)

Evaluate the surface integral $\int\int_S (ax^2 - by^2)\,dS$ where S is the cone given by the equation $x^2 + y^2 - z^2 = 0$ with $0 \le z \le 1$ and a, b are positive constants.

$x^2+y^2=z^2 \qquad z=r$

$G(r,\theta) = (r\cos\theta, r\sin\theta, r)$ ✓

$\|N\| = \sqrt{2}\,r$ ✓

$\sqrt{2}\int_0^{2\pi}\int_0^1 ar^3\cos^2\theta - br^3\sin^2\theta \,dr\,d\theta = \frac{\sqrt{2}}{4}\int_0^{2\pi} a\cos^2\theta - b\sin^2\theta\,d\theta =$

$\frac{\sqrt{2}}{4}\int_0^{2\pi} \frac{a}{2}(1+\cos(2\theta)) - \frac{b}{2}(1-\cos(2\theta))\,d\theta =$

$\frac{1}{2\sqrt{2}}\left(\frac{a}{2}(\theta + \cancel{\tfrac{1}{2}\sin(2\theta)}) - \frac{b}{2}(\theta - \cancel{\tfrac{1}{2}\sin(2\theta)})\right)\Big|_{\theta=0}^{\theta=2\pi} =$

$\frac{1}{2\sqrt{2}}\left(\frac{2\pi a}{2} - \frac{2\pi b}{2}\right) = \boxed{\frac{1}{2\sqrt{2}}(\pi a - \pi b)}$ ✓

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 - bv^2)$ with domain $D = \{(u,v) \mid u^2 + v^2 \le 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{n}$ pointing to the negative z-direction (that is the z-component of $\mathbf{n}$ is negative). Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = \langle y^2, x^2, z \rangle$.

$\frac{\partial G}{\partial u} = \langle 1, 0, 2au \rangle$ $\quad T_u \times T_v = \langle -2au, 2bv, 1 \rangle$ $\xrightarrow{\text{negative } z} \langle 2au, -2bv, -1 \rangle$

$\frac{\partial G}{\partial v} = \langle 0, 1, -2bv \rangle$ $\quad F(G(u,v)) = (v^2, u^2, au^2 - bv^2)$

$\vec{N} \cdot F(G(u,v)) \to \iint 2auv^2 - 2bvu^2 - au^2 + bv^2 \, du\, dv$

polar

$$\int_0^{2\pi}\int_0^1 2ar^4 \cos\theta \sin^2\theta - 2br^4 \sin\theta \cos^2\theta - ar^3\cos^2\theta + br^3 \sin^2\theta \, dr\, d\theta =$$

$$\int_0^{2\pi} \frac{2}{5} a \cos\theta \sin^2\theta - \frac{2}{5} b \sin\theta \cos^2\theta - \frac{a}{4}\cos^2\theta + \frac{b}{4}\sin^2\theta \, d\theta =$$

$$\int_0^{2\pi} \frac{2}{5} a \sin^2\theta \cos\theta - \frac{2}{5} b \cos^2\theta \sin\theta - \frac{a}{8}(1 + \cos(2\theta)) + \frac{b}{8}(1 - \cos(2\theta)) \, d\theta$$

$$= \cancel{\frac{2a}{5}\left(\frac{\sin^3\theta}{3}\right)} + \cancel{\frac{2b}{5}\left(\frac{\cos^3\theta}{3}\right)} - \frac{a}{8}\left(\theta + \cancel{\tfrac{1}{2}\sin(2\theta)}\right) + \frac{b}{8}\left(\theta - \tfrac{1}{2}\sin(2\theta)\right)\Big|_{\theta=0}^{2\pi}$$

$$= \frac{-a(\cancel{2}\pi)}{\cancel{8}4} + \frac{b(\cancel{2}\pi)}{\cancel{8}4} = \boxed{\frac{\pi}{4}(b - a)}$$