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MATH 32B Midterm II, Fall 2018

Name:

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

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Problem 1. (4)

Evaluate the following integral

$$\int\int_D ax^2 - by^2 dxdy.$$

Here D is the finite region in $\mathbf{R}^2$ bounded by the lines $x+y=a, x+y=-a, x-y=b, x-y=-b$, where a and b are two positive constants.

let $u = x+y$, $v = x-y$

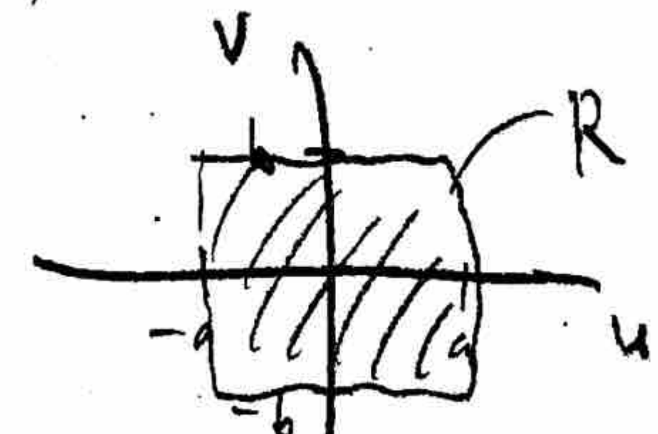
$u=a$, $u=-a$, $v=b$, $v=-b$

$u+v = 2x$

$x = \frac{u}{2} + \frac{v}{2}$ $y = x - v$

$y = \frac{u}{2} + \frac{v}{2} - v$

$y = \frac{u}{2} - \frac{v}{2}$



$-a \le u \le a$

$-b \le v \le b$

$G(u,v) = \left(\frac{u}{2}+\frac{v}{2}, \frac{u}{2}-\frac{v}{2}\right)$

$$\mathrm{Jac}(G) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = \left|\left(\tfrac{1}{2}\right)\left(-\tfrac{1}{2}\right) - \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\right|$$

$$= \left|-\tfrac{1}{4} - \tfrac{1}{4}\right| = \left|-\tfrac{2}{4}\right| = \left|-\tfrac{1}{2}\right| = \tfrac{1}{2}$$

$$\frac{1}{2}\int_{v=-b}^{b}\int_{u=-a}^{a} a\left(\frac{u}{2}+\frac{v}{2}\right)^2 - b\left(\frac{u}{2}-\frac{v}{2}\right)^2 du\,dv = \frac{1}{2}\int_{-b}^{b}\int_{-a}^{a} a\left(\frac{u^2}{4}+\frac{uv}{2}+\frac{v^2}{4}\right) - b\left(\frac{u^2}{2}-\frac{uv}{2}+\frac{v^2}{2}\right) du\,dv$$

$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{4}\right)\int_{-b}^{b}\int_{-a}^{a} a(u^2+2uv+v^2) - b(u^2-2uv+v^2)\,du\,dv = \frac{1}{8}\int_{-b}^{b}\int_{-a}^{a} au^2 \cancel{+2auv} + av^2 - bu^2 \cancel{+2buv} - bv^2 du\,dv$$

$$= \frac{1}{8}\int_{-b}^{b}\int_{-a}^{a}(a-b)u^2 + (a-b)v^2\,du\,dv = \frac{a-b}{8}\int_{-b}^{b}\int_{-a}^{a} u^2+v^2\,du\,dv$$

$$= \frac{a-b}{8}\int_{-b}^{b} \frac{u^3}{3} + uv^2 \Big|_{-a}^{a} dv = \frac{a-b}{8}\int_{-b}^{b} \frac{a^3}{3} + av^2 - \left(-\frac{a^3}{3} - av^2\right)dv = \frac{a-b}{8}\int_{-b}^{b} \frac{2a^3}{3} + 2av^2\,dv$$

$$= \frac{a-b}{8}\left[\frac{2a^3}{3}v + \frac{2}{3}av^3 \Big|_{-b}^{b}\right] = \left(\frac{a-b}{8}\right)\left[\frac{2a^3b}{3} + \frac{2}{3}ab^3 + \frac{2a^3b}{3} + \frac{2}{3}ab^3\right] = \left(\frac{a-b}{8}\right)\left(\frac{4a^3b + 4ab^3}{3}\right)$$

$$= \boxed{\left(\frac{a-b}{6}\right)(a^3b + ab^3)}$$

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Problem 2. (4) Find the line integral

$$\int_{C_1+C_2} xydx - xzdy + 2yzdz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (1,1,1)$, $Q = (1,1,0)$ and $R = (0,1,1)$.

C_1: $(1,1,1)$ to $(1,1,0)$

$r(t) = (1-t)\langle 1,1,1\rangle + t\langle 1,1,0\rangle = \langle 1-t, 1-t, 1-t\rangle + \langle t, t, 0\rangle$

$r(t) = \langle 1, 1, 1-t\rangle$ ✓

$r'(t) = \langle 0, 0, -1\rangle$

$\int_0^1 \cancel{(1)(1)(0)} - \cancel{(1)(1-t)(0)} + 2(1)(1-t)(-1)\,dt = \int_0^1 -2+2t\,dt$

$= -2t + t^2 \Big|_0^1 = -2+1 = \underline{-1}$ ✓

C_2: $(1,1,0)$ to $(0,1,1)$

$r(t) = (1-t)\langle 1,1,0\rangle + t\langle 0,1,1\rangle = \langle 1-t, 1-t, 0\rangle + \langle 0, t, t\rangle$

$r(t) = \langle 1-t, 1, t\rangle$ ✓

$r'(t) = \langle -1, 0, 1\rangle$

$\int_0^1 (1-t)(1)(-1) \cancel{-(1-t)(t)(0)} + 2(1)(t)(1)dt = \int_0^1 t-1+2t\,dt = \int_0^1 3t-1\,dt$

$= \frac{3t^2}{2} - t\Big|_0^1 = \frac{3}{2} - 1 = \frac{3}{2} - \frac{2}{2} = \underline{\frac{1}{2}}$ ✓

$\frac{1}{2} - 1 = -\frac{1}{2}$

$$\boxed{\int_{C_1+C_2} xydx - xzdy + 2yzdz = -\frac{1}{2}}$$ ✓

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Problem 3. (4)

Let $\mathbf{F} = (3x^2 - 4xyz^2)\mathbf{i} + (-2x^2z^2 + 2y)\mathbf{j} + (-4x^2yz + 1)\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide if $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot dr$, where C is given by the parametric equation $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 1$ with the orientation given by the parametrization.

(i) $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$

$-4xz^2 = -4xz^2$ ✓, $-4x^2z = -4x^2z$ ✓, $-8xyz = -8xyz$ ✓

Conservative

and defined on $\mathbb{R}^3$

(ii) $\int 3x^2 - 4xyz^2 \, dx = x^3 - 2x^2yz^2 + f(y,z)$

$\int -2x^2z^2 + 2y \, dy = -2x^2yz^2 + y^2 + g(x,z)$

$\int -4x^2yz + 1 \, dz = -2x^2yz^2 + z + h(x,y)$

$V(x,y,z) = x^3 - 2x^2yz^2 + y^2 + z + C$

(iii) $t=0: (0,0,0)$ $t=1: (1,1,1)$

$\int_C F \cdot dr = V(1,1,1) - V(0,0,0)$

$= (1) - 2(1)(1)(1) + 1 + 1 - 0$

$= 1 - 2 + 1 + 1$

$= 3 - 2$

$= \boxed{1}$

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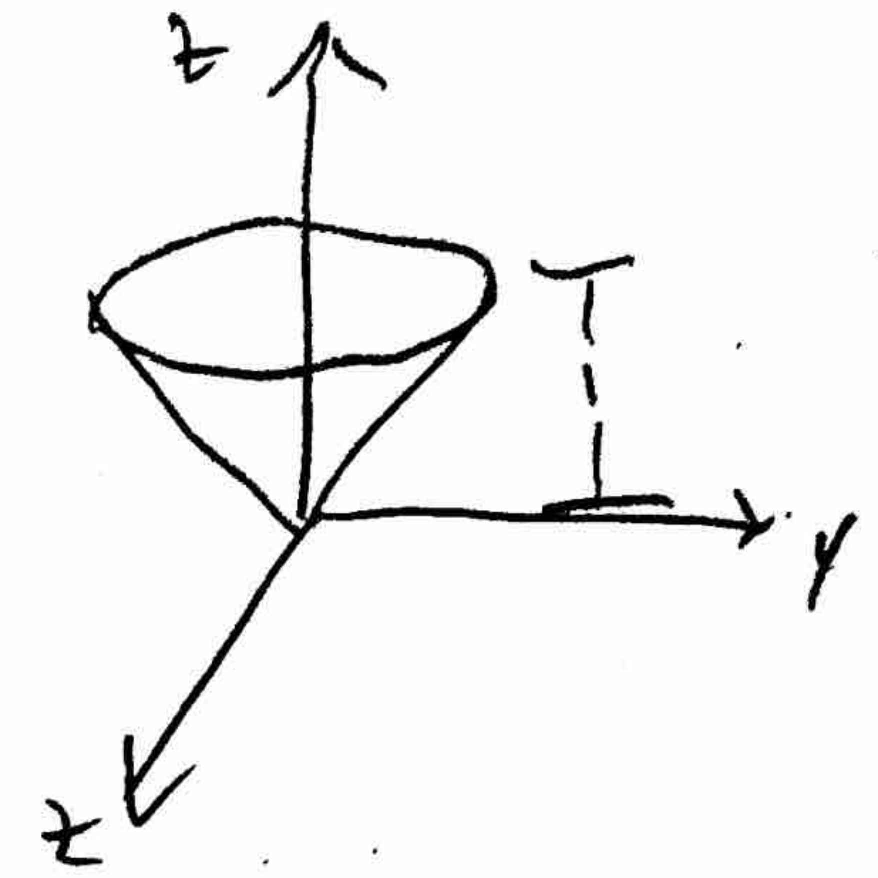

Problem 4. (4)

Evaluate the surface integral $\int\int_S (ax^2 - by^2)dS$ where S is the cone given by the equation $x^2 + y^2 - z^2 = 0$ with $0 \le z \le 1$ and a, b are positive constants.

$$r(r,\theta) = \langle r\cos\theta, r\sin\theta, r\rangle \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$$

$$T_r = \langle \cos\theta, \sin\theta, 1\rangle \qquad T_\theta = \langle -r\sin\theta, r\cos\theta, 0\rangle$$

$$T_r \times T_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta)\vec{i} - (r\sin\theta)\vec{j} + (r\cos^2\theta + r\sin^2\theta)\vec{k}$$

$$= \langle -r\cos\theta, -r\sin\theta, r\rangle$$

$$\|T_r \times T_\theta\| = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2} = \sqrt{2r^2} = \sqrt{2}\, r$$

$$\|T_r \times T_\theta\| = \sqrt{2}\, r$$

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}(ar^2\cos^2\theta - br^2\sin^2\theta)\sqrt{2}\, r \, dr\, d\theta = \sqrt{2}\int_0^{2\pi}\int_0^1 ar^3\cos^2\theta - br^3\sin^2\theta \, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{ar^4}{4}\cos^2\theta - \frac{br^4}{4}\sin^2\theta \Big|_0^1 d\theta = \sqrt{2}\int_0^{2\pi} \frac{a}{4}\cos^2\theta - \frac{b}{4}\sin^2\theta \, d\theta$$

$$= \frac{\sqrt{2}}{4}\int_0^{2\pi} a\cos^2\theta - b\sin^2\theta \, d\theta = \frac{\sqrt{2}}{4}\int_0^{2\pi} a\left(\frac{1+\cos 2\theta}{2}\right) - b\left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{\sqrt{2}}{8}\int_0^{2\pi} a(1+\cos 2\theta) - b(1-\cos 2\theta)\, d\theta = \frac{\sqrt{2}}{8}\int_0^{2\pi} a + a\cos 2\theta - b + b\cos 2\theta \, d\theta$$

$$= \frac{\sqrt{2}}{8}\int_0^{2\pi} (a-b) + (a+b)\cos 2\theta \, d\theta = \frac{\sqrt{2}}{8}\left[(a-b)\theta + \frac{(a+b)}{2}\sin 2\theta \Big|_0^{2\pi}\right]$$

$$= \frac{\sqrt{2}}{8}\left[(a-b)(2\pi)\right] = \boxed{\frac{\sqrt{2}}{4}(a-b)\pi} \quad ✓$$

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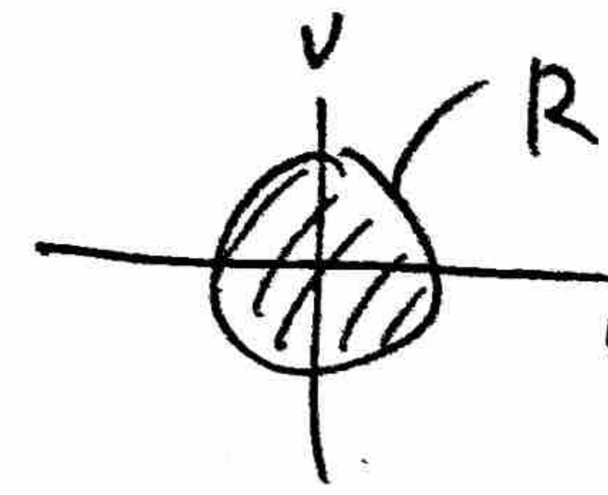
Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, au^2 - bv^2)$ with domain $D = \{(u,v) \mid u^2 + v^2 \le 1\}$ where a and b are two constants. Orient S with the normal vector field $\mathbf{n}$ pointing to the negative z-direction (that is the z-component of $\mathbf{n}$ is negative). Find

$$\iint_S \mathbf{F} \cdot d\mathbf{S}.$$



Here the vector field $\mathbf{F} = \langle y^2, x^2, z \rangle$.

$\vec{T}_u = \langle 1, 0, 2au \rangle \qquad \vec{T}_v = \langle 0, 1, -2bv \rangle$

$$\vec{T}_u \times \vec{T}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2au \\ 0 & 1 & -2bv \end{vmatrix} = (-2au)\vec{i} - (-2bv)\vec{j} + (1)\vec{k}$$

$$= \langle -2au, 2bv, 1 \rangle$$

$\underline{\vec{n} = \langle 2au, -2bv, -1 \rangle}$ ✓

$u = r\cos\theta$

$v = r\sin\theta$

$u^2 + v^2 = r^2$

$$\iint_R \langle v^2, u^2, au^2 - bv^2 \rangle \cdot \langle 2au, -2bv, -1 \rangle \, dA$$

$$\iint_R 2auv^2 - 2bu^2v - au^2 + bv^2 \, dA$$

$$\int_0^{2\pi}\int_0^1 \left(\cancel{2a(r\cos\theta)(r^2\sin^2\theta) - 2b(r^2\cos^2\theta)(r\sin\theta)} - a(r^2\cos^2\theta) + b(r^2\sin^2\theta)\right) r\,dr\,d\theta$$

$$\int_0^{2\pi}\int_0^1 br^3\sin^2\theta - ar^3\cos^2\theta \, dr\,d\theta = \int_0^{2\pi} \frac{br^4}{4}\sin^2\theta - \frac{ar^4}{4}\cos^2\theta \Big|_0^1 \, d\theta$$

$$= \int_0^{2\pi} \frac{b}{4}\sin^2\theta - \frac{a}{4}\cos^2\theta \, d\theta = \frac{1}{4}\int_0^{2\pi} b\sin^2\theta - a\cos^2\theta \, d\theta$$

$$\frac{1}{4}\int_0^{2\pi} b\left(\frac{1-\cos 2\theta}{2}\right) - a\left(\frac{1+\cos 2\theta}{2}\right) d\theta = \frac{1}{8}\int_0^{2\pi} b - b\cos 2\theta - a - a\cos 2\theta \, d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} (b-a) - b\cos 2\theta - a\cos 2\theta \, d\theta = \frac{1}{8}\left[(b-a)\theta - \frac{b\sin 2\theta}{2} - \frac{a\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{8}\left[(b-a)(2\pi)\right] = \boxed{\frac{\pi}{4}(b-a)}$$ ✓