1	4
2	3.5
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MATH 32B Midterm II, Fall 2015

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the line integral

$$\int_{C_1+C_2} xzdx + xydy + y^2dz,$$

where C_1 is the line segment connecting P and Q with the orientation from P to Q, and C_2 is the line segment connecting Q and R with the orientation from Q to R. Here $P = (0,1,1)$, $Q = (1,0,1)$ and $R = (1,1,0)$.

$r_1(t) = (1-t)\langle 0,1,1\rangle + t\langle 1,0,1\rangle = \langle t, 1-t, 1\rangle, \quad 0 \le t \le 1$

$r_1'(t) = \langle 1,-1,0\rangle \quad F(r_1(t)) = \langle t, t-t^2, 1-2t+t^2\rangle$

$F(r_1(t)) \cdot r_1'(t) = \cancel{t} - \cancel{t} + t^2 \Rightarrow \int_{C_1} F \cdot ds = \int_0^1 t^2 dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3}$

$r_2(t) = (1-t)\langle 1,0,1\rangle + t\langle 1,1,0\rangle = \langle 1, t, 1-t\rangle, \quad 0 \le t \le 1$

$r_2'(t) = \langle 0, 1, -1\rangle \quad F(r_2(t)) = \langle 1-t, t, t^2\rangle$

$F(r_2(t)) \cdot r_2'(t) = t - t^2 \Rightarrow \int_{C_2} F \cdot ds = \int_0^1 t - t^2 dt = \frac{1}{2}t^2 - \frac{1}{3}t^3\Big|_0^1$

$= \frac{1}{2} - \frac{1}{3}$

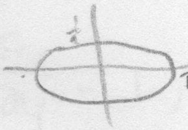
$\int_{C_1+C_2} F \cdot ds = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} = \frac{1}{2}$

Problem 2. (4)

Evaluate the following integral

$$\int\int_D \sin(x^2/a^2 + y^2/b^2)dxdy.$$

Here D is the region in $\mathbf{R}^2$ bounded by the ellipse $b^2x^2 + a^2y^2 = 1$, where a and b are two positive constants.



let $u = bx,\ v = ay \quad u^2+v^2=1$

$$F = \sin\left(\frac{u^2}{a^2b^2} + \frac{v^2}{a^2b^2}\right) = \sin\left(\frac{u^2+v^2}{a^2b^2}\right) \qquad \text{Jac}(G) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} b & 0 \\ 0 & a \end{vmatrix}$$

$$= ab$$

Now let $r = \sqrt{u^2+v^2} \quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ \text{Jac}(G) = rab$

$$F = \sin\left(\frac{r^2}{a^2b^2}\right)$$

$$I = \int_0^{2\pi}\int_0^1 \sin\left(\frac{r^2}{a^2b^2}\right) rab\, dr\, d\theta$$

$$= 2\pi ab(a^2b^2)\left(-\cos\left(\frac{r^2}{a^2b^2}\right)\cdot\frac{1}{2}\right)\Big|_0^1$$

-0.5

$$= \pi a^3 b^3\left(-\cos\left(\frac{1}{a^2b^2}\right) + \cos 0\right)$$

$$= \pi ab = \pi ab^3\left(1 - \cos\left(\frac{1}{a^2b^2}\right)\right)$$

2.5

Problem 3. (4)

Let $\mathbf{F} = y\mathbf{i} + (x + z + ze^{yz})\mathbf{j} + (y + z^3 + ye^{yz})\mathbf{k}$ defined on $\mathbf{R}^3$.
(i) Decide whether or not $\mathbf{F}$ is conservative.
(ii) If $\mathbf{F}$ is conservative, find the potential function V, such that $\mathbf{F} = \nabla V$.
(iii) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is given by the parametric equation $x = t^3,\ y = t,\ z = t^3,\ 0 \le t \le 1$ with the orientation given by the parametrization.

$$\frac{\partial F_1}{\partial y} = 1, \quad \frac{\partial F_2}{\partial x} = 1, \quad \frac{\partial F_2}{\partial z} = 1 + yze^{yz}, \quad \frac{\partial F_3}{\partial y} = 1 + yze^{yz}$$

$$\frac{\partial F_1}{\partial z} = 0, \quad \frac{\partial F_3}{\partial x} = 0 \Rightarrow F \text{ conservative}$$

✓

$$V = \int y\,dx = xy + c(y,z) \qquad U = \int x + z + ze^{yz}\,dy = (x+z)y + \frac{z}{y}e^{yz} + d(x,z)$$

$$V = \int y + z^3 + ye^{yz}\,dz = yz + \tfrac{1}{4}z^4 + \frac{y}{z}e^{yz} + g(x,y)$$

$$\therefore V = (x+z)y + \cancel{\left(\frac{y}{z} + \frac{z}{y}\right)}e^{yz} + \frac{1}{4}z^4 + C$$

$$P = r(0) = (0,0,0), \quad Q = r(1) = (1,1,1)$$

$$\int_C F \cdot dr = V(Q) - U(P) \qquad V(t) = (t^3 + t^3)t + \left(\frac{t}{t^2} + t^2\right)e^{t^4} + \frac{1}{4}(t^3)^4$$

$$= 2 + \cancel{2e + \frac{1}{4}}$$

Problem 4. (4)

Evaluate the surface integral $\int\int_S (2x^2 - 3z^2)dS$ where S is the sphere given by the equation $x^2 + y^2 + z^2 = a^2$.

$G(\varphi, \theta) = (a\sin\varphi\cos\theta,\ a\sin\varphi\sin\theta,\ a\cos\varphi)$ $0 \le \theta \le 2\pi,\ 0 \le \varphi \le \pi$

Sphere $\Rightarrow \vec{n} = a^2\sin\varphi\,\vec{e_r}$ so $\|\vec{n}\| = a^2\sin\varphi$ ✓

$F = 2a^2\sin^2\varphi\cos^2\varphi - 3a^2\cos^2\varphi$

$\iint_S F\,dS = \int_0^{2\pi}\int_0^{\pi} 2a^4\sin^3\varphi\cos^2\varphi - 3a^3\sin\varphi\cos^2\varphi\,d\varphi\,d\theta$

$= 2\pi\left[\int_0^{\pi} 2a^4\sin\varphi\cos^2\varphi(1-\cos^2\varphi)\,d\varphi - \int_0^{\pi} 3a^3\sin\varphi\cos^3\varphi\,d\varphi\right]$

$= 2\pi\left[2a^4\left(-\frac{1}{3}\cos^3\varphi + \frac{1}{5}\cos^5\varphi\right)\Big|_0^{\pi} - 3a^3\left(\frac{1}{-4}\cos^4\varphi\right)\Big|_0^{\pi}\right]$

$= 2\pi\left[2a^4\left(\frac{1}{3} - \frac{1}{5} - \left(-\frac{1}{3} + \frac{1}{5}\right)\right) - \cancel{3a^3\left(-\frac{1}{4} + \frac{1}{4}\right)}\right]$

$= 4\pi a^4\left(\frac{2}{3} - \frac{2}{5}\right)$

Problem 5. (4)

Let S be a surface given by the parametric equation $G(u,v) = (u, v, u^2 + 2v^2)$ with domain $D = \{(u,v) | \, u^2 + v^2 \leq 1\}$. Orient S with the normal pointing to the negative z-direction. Find the flux

$$\int\int_S \mathbf{F} \cdot d\mathbf{S}.$$

Here the vector field $\mathbf{F} = < z, x^2, y^2 >$.

$T_u = \langle 1, 0, 2u \rangle \quad T_v = \langle 0, 1, 4v \rangle$

$\vec{n} = T_u \times T_v = \hat{k} - 4v\hat{j} + 2u\hat{i} = \langle 2u, -4v, 1 \rangle$

$-z$ dir $\Rightarrow \vec{n} = \langle -2u, 4v, -1 \rangle$

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$F(u,v) = \langle u^2 + 2v^2, u^2, v^2 \rangle$

$F(u,v) \cdot \vec{n} = -2u^2 - \cancel{4uv^2} + \cancel{4uv^2} - v^2 = -2u^2 - v^2$

$D: u^2 + v^2 \leq 1 \Rightarrow 0 \leq r \leq 1, \; 0 \leq \theta \leq 2\pi \quad u = r\cos\theta, \; v = r\sin\theta$

$F = -(2r^2\cos^2\theta + r^2\sin^2\theta) = -r^2(\cos^2\theta + 1)$

$\iint_S F \cdot dS = \int_0^{2\pi} \cos^2\theta + 1 \, d\theta \int_0^1 -r^3 \, dr$ ✓

$= -\frac{1}{4} r^4 \Big|_0^1 \int_0^{2\pi} \frac{1}{2}(1 + \cos 2\theta) \, d\theta$

$= -\frac{1}{4} \cdot \frac{1}{2} \left(\theta + \frac{1}{2}\sin 2\theta\right) \Big|_0^{2\pi}$

$= -\frac{1}{8}(2\pi)$

$= -\frac{\pi}{4}$