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MATH 32B Midterm I, Winter 2019

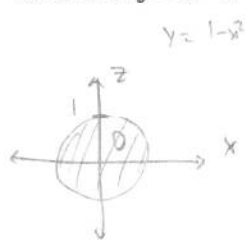
Name:

Circle Your TA's Name and Section Number: Eli Sadovnik 2A
 2B, Ben Szczesny 2C, 2D, Victoria Kala 2E 2F

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral $\iiint_E x^2 dx dy dz$. Here E is the finite solid bounded by the surface $y = 1 - x^2 - z^2$ and the plane $y = 0$.



$y = 1 - x^2 - z^2$ and $y = 0$ intersect at $x^2 + z^2 = 1$

→ Use cylindrical coordinates

$y = y$ $z = r \sin \theta$ $x = r \cos \theta$

$0 \leq y \leq 1 - x^2 - z^2 \Rightarrow 0 \leq y \leq 1 - r^2$

$E = \{(r, \theta, y) : 0 \leq r \leq 1, 0 \leq \theta < 2\pi, 0 \leq y \leq 1 - r^2\}$

$x^2 = r^2 \cos^2 \theta$

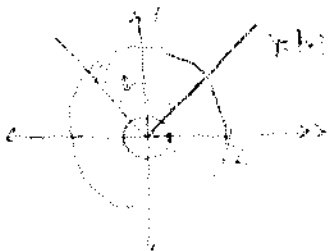
$$\begin{aligned} \iiint_E x^2 dx dy dz &= \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} (r^2 \cos^2 \theta) (r dy dr d\theta) \\ &= \int_0^{2\pi} \int_0^1 (r^3 \cos^2 \theta) y \Big|_{y=0}^{1-r^2} dr d\theta = \int_0^{2\pi} \int_0^1 (r^3 - r^5) \cos^2 \theta dr d\theta \\ &= \int_0^{2\pi} \left(\frac{r^4}{4} - \frac{r^6}{6} \right) \cos^2 \theta \Big|_{r=0}^1 d\theta = \int_0^{2\pi} \left(\frac{1}{4} - \frac{1}{6} \right) \cos^2 \theta d\theta \\ &= \frac{1}{12} \int_0^{2\pi} \left(\frac{1}{2} (2 \cos^2 \theta - 1) + \frac{1}{2} \right) d\theta = \frac{1}{24} \int_0^{2\pi} (\cos 2\theta + 1) d\theta \end{aligned}$$

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$$= \frac{1}{24} \left(\frac{1}{2} \sin(2\theta) + \theta \right) \Big|_{\theta=0}^{2\pi} = \frac{1}{24} \left(\frac{1}{2} \sin(4\pi) - \frac{1}{2} \sin(0) + 2\pi \right) = \frac{\pi}{12}$$

Problem 2. (4)

Compute the center of mass of $D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4, y \geq |x|\}$. Here the density function $\rho(x, y) = y^2$.



Use polar coordinates
 $x = r \cos \theta, y = r \sin \theta$
 plane element $r^2 \sin^2 \theta$
 $1 \leq r \leq 2$
 $y = |x| \Rightarrow r \sin \theta = r \cos \theta$

$$\Rightarrow \frac{\sin \theta}{\cos \theta} = 1$$

Since D is in the first quadrant, $\theta \in [0, \pi/2]$

$$\Rightarrow |\tan \theta| = 1$$

$$\Rightarrow \tan \theta = 1$$

$$\Rightarrow \theta \in [0, \pi/4], [\pi/4, \pi/2]$$

$$\text{Check for } D \quad \pi/4 \leq \theta \leq 3\pi/4$$

$\rho(r, \theta) = y^2 = r^2 \sin^2 \theta$

$$M_x = \iint_D y \rho(x, y) dx dy = \int_{\pi/4}^{3\pi/4} \int_1^2 (r \sin \theta) (r^2 \sin^2 \theta) r dr d\theta = \int_{\pi/4}^{3\pi/4} \left[\frac{r^4}{4} \sin^3 \theta \right]_{r=1}^{r=2} d\theta = \int_{\pi/4}^{3\pi/4} \left(\frac{16-1}{4} \right) \sin^3 \theta d\theta$$

$$= \frac{15}{4} \int_{\pi/4}^{3\pi/4} (\sin \theta)(1 - \cos^2 \theta) d\theta = \frac{15}{8} \int_{\pi/4}^{3\pi/4} (-\cos \theta d\cos \theta + 2 \sin \theta) = \frac{15}{8} \left[\frac{1}{2} \cos^2 \theta + \theta \right]_{\pi/4}^{3\pi/4}$$

$$= \frac{15}{8} \left(\frac{1}{2} \cos^2 \frac{3\pi}{4} + \frac{3\pi}{4} - \left(\frac{1}{2} \cos^2 \frac{\pi}{4} + \frac{\pi}{4} \right) \right) = \frac{15}{8} \left(1 + \frac{\pi}{2} \right)$$

Since region D is symmetric about the y-axis and $\rho(x, y) = y^2 = f(y)$ is also symmetric about the y-axis, $x_{cm} = 0$.

$$y_{cm} = \frac{1}{M} \iint_D y \rho(x, y) dx dy = \frac{1}{M} \int_{\pi/4}^{3\pi/4} \int_1^2 (r \sin \theta) (r^2 \sin^2 \theta) r dr d\theta = \frac{15}{8} \int_{\pi/4}^{3\pi/4} \left(\frac{16-1}{4} \right) \sin^3 \theta d\theta$$

Use $\int \sin^3 \theta d\theta = -\cos \theta + \frac{1}{3} \cos^3 \theta$

$$= \frac{15}{8} \int_{\pi/4}^{3\pi/4} \sin^2 \theta (-\cos \theta) d\theta = \frac{15}{8} \int_{\pi/4}^{3\pi/4} (1 - \cos^2 \theta) (-\cos \theta) d\theta = \frac{15}{8} \left[-\cos \theta + \frac{\cos^3 \theta}{3} \right]_{\pi/4}^{3\pi/4}$$

$$= \frac{15}{8} \left(\cos \frac{\pi}{4} - \frac{\cos^3 \frac{\pi}{4}}{3} - \left(\cos \frac{3\pi}{4} - \frac{\cos^3 \frac{3\pi}{4}}{3} \right) \right) = \frac{15}{8} \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} \right) \right) = \frac{15\sqrt{2}}{8} \left(1 - \frac{1}{6} \right) = \frac{31\sqrt{2}}{6 \cdot 8} = \frac{31\sqrt{2}}{48} = \frac{31\sqrt{2}}{6 \left(\frac{15}{8} (1 + \frac{\pi}{2}) \right)} = \frac{124\sqrt{2}}{45(1 + \frac{\pi}{2})}$$

\Rightarrow Center of mass

$$\left(0, \frac{124\sqrt{2}}{45(1 + \frac{\pi}{2})} \right)$$

Problem 3. (4)

Find the iterated integral $\int_0^R \int_0^{\sqrt{R^2-x^2}} \cos^2((x^2+y^2)) dy dx$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.



$(x, y) \in D \iff (x, y) \in [0, R] \times [0, \sqrt{R^2-x^2}]$
 $\iff (x, y) \in D_1 \cup D_2$
 $D_1 = \{(x, y) \mid 0 \leq x \leq R, 0 \leq y \leq x\}$
 $D_2 = \{(x, y) \mid 0 \leq x \leq R, x \leq y \leq \sqrt{R^2-x^2}\}$
 $\cos^2(x^2+y^2) = \cos^2(r^2)$

$$\begin{aligned}
 \int_0^R \int_0^{\sqrt{R^2-x^2}} \cos^2(x^2+y^2) dy dx &= \iint_D \cos^2(x^2+y^2) dx dy \\
 &= \iint_{D_1} \cos^2(x^2+y^2) dx dy + \iint_{D_2} \cos^2(x^2+y^2) dx dy \\
 &= \frac{1}{4} \int_0^{R^2} \int_0^{\sqrt{u}} (\cos^2(u) + 1) du dv + \frac{1}{4} \int_0^{R^2} \int_0^{\sqrt{u}} (\cos^2(u) + 1) du dv \\
 &= \frac{1}{4} \int_0^{R^2} (\cos^2(u) + 1) \frac{1}{2} du + \frac{1}{4} \int_0^{R^2} (\cos^2(u) + 1) \frac{1}{2} du \\
 &= \left(\frac{1}{8} + \frac{1}{4} \right) \int_0^{R^2} (\cos^2(u) + 1) du = \frac{3}{8} \int_0^{R^2} (\cos^2(u) + 1) du
 \end{aligned}$$

$u = r^2$
 $dv = 2r dr$

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Problem 4. (4)

Find

$$\iiint_E x^2 + y^2 dx dy dz,$$

where E is the finite solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Note: E is inside both the sphere and the cone.



Use spherical coordinates
 $\rho = \sqrt{x^2 + y^2 + z^2}$
 $\phi = \arccos(z/\rho)$
 $\theta = \arctan(y/x)$
 $z = \rho \cos \phi$
 $x = \rho \sin \phi \cos \theta$
 $y = \rho \sin \phi \sin \theta$
 $dx dy dz = \rho^2 \sin \phi d\rho d\phi d\theta$
 $\rho = 1$
 $\phi = \pi/4$
 $\theta = 0$ to 2π

$$\iiint_E (x^2 + y^2) dx dy dz = \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 (\rho^2 \sin^2 \phi) \rho^2 \sin \phi d\rho d\phi d\theta$$

Use $\cos^2 \phi = \frac{1 + \cos 2\phi}{2}$

$$= \int_0^{2\pi} \int_0^{\pi/4} \int_0^1 \rho^4 \sin^2 \phi d\rho d\phi d\theta = \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{5} \rho^5 \sin^2 \phi d\phi d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \int_0^{\pi/4} (1 + \cos 2\phi) \sin^2 \phi d\phi d\theta$$

$$= \frac{1}{5} \int_0^{2\pi} \left[-\frac{1}{3} \cos 2\phi + \frac{1}{5} \cos 4\phi + \frac{1}{5} \right] d\phi d\theta = \frac{1}{5} \left[\frac{2\pi}{5} - \frac{2\pi}{15} \right] = \frac{2\pi}{15}$$

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{\sin(x^2+y^2) \cdot \sin z} dV,$$

where E is the solid inside the cylinder $x^2 + y^2 = \pi/4$ with $0 \leq z \leq \pi/4$.

$$0 \leq x^2 + y^2 \leq \frac{\pi}{4}$$

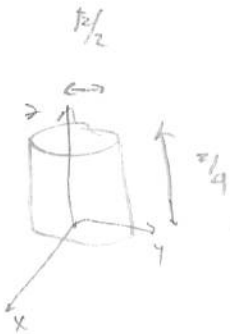
$$\Rightarrow \sin(0) \leq \sin(x^2+y^2) \leq \sin\left(\frac{\pi}{4}\right) \Rightarrow 0 \leq \sin(x^2+y^2) \leq \frac{1}{\sqrt{2}}$$

$$0 \leq z \leq \frac{\pi}{4}$$

$$\Rightarrow \sin(0) \leq \sin(z) \leq \sin\left(\frac{\pi}{4}\right) \Rightarrow 0 \leq \sin(z) \leq \frac{1}{\sqrt{2}}$$

$$\Rightarrow 0 \leq \sin(x^2+y^2) \sin(z) \leq \frac{1}{2}$$

$$\iiint_E e^0 dV \leq \iiint_E e^{\sin(x^2+y^2) \sin(z)} dV \leq \iiint_E e^{1/2} dV$$



$$\text{Volume}(E) \leq \iiint_E e^{\sin(x^2+y^2) \sin(z)} dV \leq \sqrt{e} \cdot \text{Volume}(E)$$

$$\text{Volume}(E) = \frac{\pi}{4} \left(2 \left(\frac{\pi}{4} \right)^2 \right) = \frac{\pi^3}{16}$$

looks like
the ~~pi~~ π^2 !

$$\frac{\pi^3}{16} \leq \iiint_E e^{\sin(x^2+y^2) \sin(z)} dV \leq \frac{\pi^3 \sqrt{e}}{16}$$

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