1	4
2	4
3	4
4	3.5
5	4
T	19.5

MATH 32B Midterm I, Winter 2019

Name:

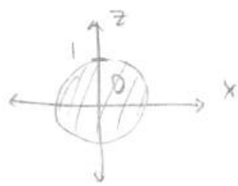
Circle Your TA's Name and Section Number: Eli Sadovnik 2A 2B, Ben Szczesny 2C 2D, Victoria Kala 2E 2F

Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral $\int\int\int_E x^2 dxdydz$. Here E is the finite solid bounded by the surface $y = 1 - x^2 - z^2$ and the plane $y = 0$.

$y = 1 - x^2 - z^2$ and $y = 0$ intersect at $x^2 + z^2 = 1$



→ Use cylindrical coordinates

$y = y \quad z = r\sin\theta \quad x = r\cos\theta$

$0 \le y \le 1 - x^2 - z^2 \Rightarrow 0 \le y \le 1 - r^2$

$E = \{(r,\theta,y): 0 \le r \le 1,\ 0 \le \theta \le 2\pi,\ 0 \le y \le 1 - r^2\}$

$x^2 = r^2\cos^2\theta$

$$\iiint_E x^2\,dxdydz = \int_0^{2\pi}\int_0^1\int_0^{1-r^2} (r^2\cos^2\theta)(r\,dy\,dr\,d\theta)$$

$$= \int_0^{2\pi}\int_0^1 (r^3\cos^2\theta)\,y\Big|_{y=0}^{1-r^2} dr\,d\theta = \int_0^{2\pi}\int_0^1 (r^3 - r^5)\cos^2\theta\,dr\,d\theta$$

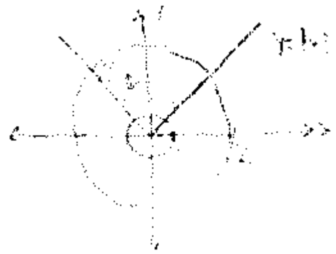
$$= \int_0^{2\pi}\left(\frac{r^4}{4} - \frac{r^6}{6}\right)\cos^2\theta\Big|_{r=0}^{1} d\theta = \int_0^{2\pi}\left(\frac{1}{4} - \frac{1}{6}\right)\cos^2\theta\,d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}\left(\frac{1}{2}(2\cos^2\theta - 1) + \frac{1}{2}\right)d\theta = \frac{1}{24}\int_0^{2\pi}(\cos(2\theta) + 1)\,d\theta$$

$$= \frac{1}{24}\left(\frac{1}{2}\sin(2\theta) + \theta\right)\Big|_{\theta=0}^{2\pi} = \frac{1}{24}\left(\frac{1}{2}\sin(4\pi) - \frac{1}{2}\sin(0) + 2\pi\right) = \boxed{\pi/12}$$

Problem 2. (4)

Compute the center of mass of $D = \{(x,y) \mid 1 \le x^2+y^2 \le 4,\ y \ge |x|\}$. Here the density function $\rho(x,y) = y^2$.



Use polar coordinates

$x = r\cos\theta$, $y = r\sin\theta$

$\rho(r\cos\theta, r\sin\theta) = r^2\sin^2\theta$

$1 \le x^2+y^2 \le 4 \Rightarrow 1 \le r \le 2$

$y = |x| \Rightarrow r\sin\theta = |r\cos\theta|$

$\Rightarrow \frac{\sin\theta}{|\cos\theta|} = 1$

$\Rightarrow |\tan\theta| = 1$

$\Rightarrow \tan\theta = \pm 1$

$\Rightarrow \theta = \ldots, \pi/4, 3\pi/4, 5\pi/4, 7\pi/4, \ldots$

Clearly for D $\pi/4 \le \theta \le 3\pi/4$

$D = \{(r,\theta) : 1 \le r \le 2,\ \pi/4 \le \theta \le 3\pi/4\}$

$$M = \iint_D \rho\, dA = \int_{\pi/4}^{3\pi/4}\int_1^2 (r^2\sin^2\theta)\, r\, dr\, d\theta = \int_{\pi/4}^{3\pi/4} \frac{r^4}{4}(\sin^2\theta)\Big|_{r=1}^{2} d\theta = \int_{\pi/4}^{3\pi/4}\left(\frac{16-1}{4}\right)\sin^2\theta\, d\theta$$

$$= \frac{15}{4}\int_{\pi/4}^{3\pi/4}\left(-\frac{1}{2}\cos(2\theta) + \frac{1}{2}\right)d\theta = \frac{15}{8}\int_{\pi/4}^{3\pi/4}(-\cos(2\theta) + 1)\,d\theta = \frac{15}{8}\left[-\frac{1}{2}\sin(2\theta) + \theta\right]_{\theta=\pi/4}^{3\pi/4}$$

$$= \frac{15}{8}\left(-\frac{1}{2}\sin\left(\frac{3\pi}{2}\right) + \frac{1}{2}\sin\left(\frac{\pi}{2}\right) + \frac{\pi}{2}\right) = \frac{15}{8}\left(1 + \frac{\pi}{2}\right)$$

Since region D is symmetric ... and $\rho(-x,y) = y^2 = \rho(x,y)$ is also symmetric ... $x_{cm} = 0$.

$$y_{cm} = \frac{1}{M}\iint_D y\rho\, dA = \frac{1}{M}\int_{\pi/4}^{3\pi/4}\int_1^2 (r^3\sin^3\theta)\, r\, dr\, d\theta = \frac{1}{M}\int_{\pi/4}^{3\pi/4} \frac{r^5}{5}\sin^3\theta\Big|_{r=1}^{2} d\theta = \frac{1}{M}\int_{\pi/4}^{3\pi/4}\left(\frac{31}{5}\right)\sin^3\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= \frac{31}{5M}\int_{\pi/4}^{3\pi/4} \sin\theta(1-\cos^2\theta)\, d\theta = \ldots = \frac{31}{5M}\left(-u + \frac{u^3}{3}\right)\Big|^{-\sqrt{2}/2}$$

$$= \frac{31}{5M}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - \frac{\sqrt{2}}{12}\right) = \frac{31\sqrt{2}}{5M}\left(1 - \frac{1}{6}\right) = \frac{31\sqrt{2}}{6M} = \frac{31\sqrt{2}}{6\left(\frac{15}{8}\left(1+\frac{\pi}{2}\right)\right)} = \frac{124\sqrt{2}}{45\left(1+\frac{\pi}{2}\right)}$$

$\Rightarrow$ Center of mass $= \boxed{\left(0, \frac{124\sqrt{2}}{45\left(1+\frac{\pi}{2}\right)}\right)}$

Problem 3. (4)

Find the iterated integral $\int_0^R \int_0^{\sqrt{R^2-x^2}} cos^2((x^2+y^2))dydx$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.

$u = r^2$

$du = 2r\,dr$

3.5

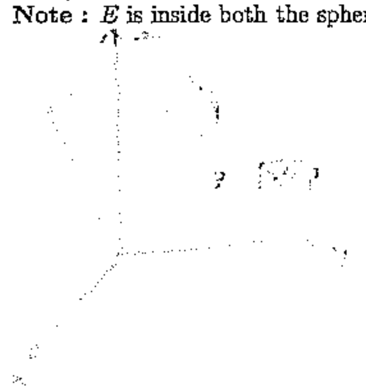
Problem 4. (4)

Find

$$\int\int\int_E x^2 + y^2 dxdydz,$$

where E is the finite solid bounded by the sphere $x^2 + y^2 + z^2 = 1$ and the cone $z = \sqrt{x^2 + y^2}$.

Note : E is inside both the sphere and the cone.



$u = \cos \phi$
$du = -\sin \phi \, d\phi$

Problem 5. (4)

Estimate the following integral

$$\int\int\int_E e^{\sin(x^2+y^2)\cdot \sin z}\,dV,$$

where E is the solid inside the cylinder $x^2+y^2=\pi/4$ with $0 \le z \le \pi/4$.

$0 \le x^2+y^2 \le \frac{\pi}{4}$

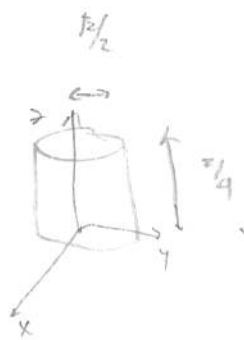
$\Rightarrow \sin(0) \le \sin(x^2+y^2) \le \sin\left(\frac{\pi}{4}\right) \Rightarrow 0 \le \sin(x^2+y^2) \le \frac{1}{\sqrt{2}}$

$0 \le z \le \frac{\pi}{4}$

$\Rightarrow \sin(0) \le \sin(z) \le \sin\left(\frac{\pi}{4}\right) \Rightarrow 0 \le \sin(z) \le \frac{1}{\sqrt{2}}$

$\Rightarrow 0 \le \sin(x^2+y^2)\sin(z) \le \frac{1}{2}$

$\iiint_E e^0\,dV \le \iiint_E e^{\sin(x^2+y^2)\sin(z)}\,dV \le \iiint_E e^{1/2}\,dV$



$\text{Volume}(E) \le \iiint_E e^{\sin(x^2+y^2)\sin(z)}\,dV \le \sqrt{e}\cdot\text{Volume}(E)$

$\text{Volume}(E) = \frac{\pi}{4}\left(\pi\left(\sqrt{\frac{\pi}{4}}\right)^2\right) = \frac{\pi^3}{16}$

looks like tho a π 2!

$$\boxed{\frac{\pi^3}{16} \le \iiint_E e^{\sin(x^2+y^2)\sin(z)}\,dV \le \frac{\pi^3\sqrt{e}}{16}}$$

4