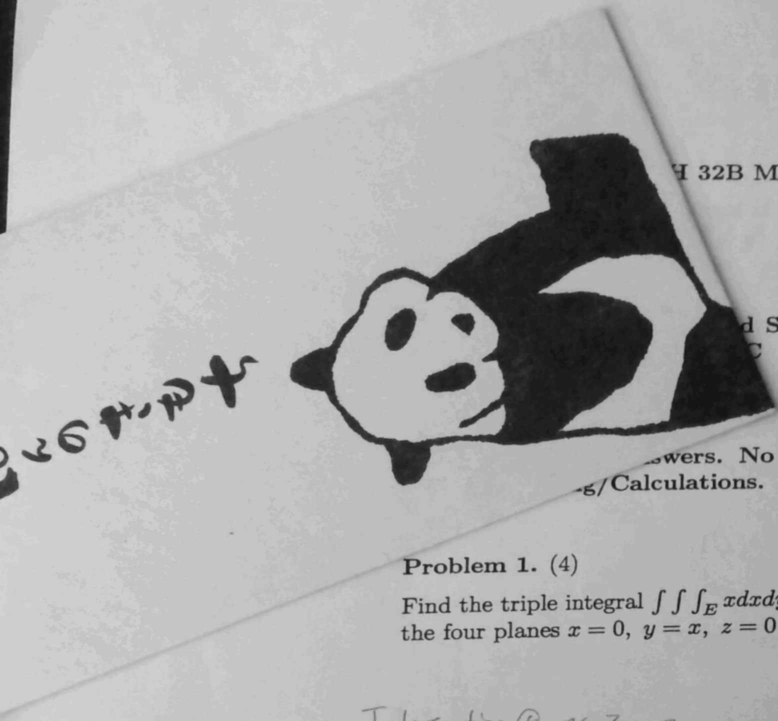
1	3
2	2
3	4
4	2.5
5	3.5
T	15



H 32B Midterm I, Winter 2017

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...swers. No Points Will Be Given Without Suffi-
...g/Calculations.

Problem 1. (4)

Find the triple integral $\int\int\int_E xdxdydz$. Here E is the finite region bounded by the four planes $x = 0,\ y = x,\ z = 0$ and $x + y + z = 2$.

$y = 2 - x - z$
$0 \le z \le 2 - 2x$
$0 \le x \le 1$

Intersection @ $x = 2 - x - z$
$z = 2 - 2x$

$z = 2 - 2x$

$$E = \int_0^1 \int_0^{2-2x} \int_x^{2-x-z} x\,dy\,dz\,dx = \int_0^1 \int_0^{2-2x} x(y|_0^{2-x-z})\,dz\,dx$$

$$= \int_0^1 \int_0^{2-2x} (2x - x^2 - xz)\,dz\,dx = \int_0^1 \left((2x - x^2)z - \frac{xz^2}{2}\Big|_0^{2-2x}\right)dx$$

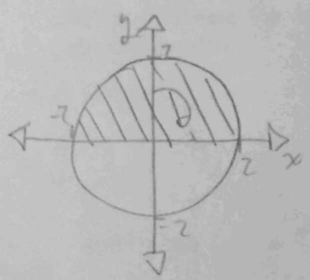
$$= \int_0^1 \left((2x - x^2)(2 - 2x) - \frac{x}{2}(2 - 2x)^2\right)dx = \int_0^1 (2x - 4x^2)\,dx$$

$$= \left(\frac{2x^2}{2} - \frac{4x^3}{3}\right)\Big|_0^1 = \left(1 - \frac{4}{3}\right)$$

$2\,4x - 4x^2 - 2x^2 + 2x^3$
$-\frac{x}{2}(4 - 4x^2 + 4x^2)$
4
$= 2x + 2x^2 - 2x^3$

Problem 2. (4)

Compute the center of of mass of $D = \{(x, y) |\ x^2 + y^2 \leq 4,\ y \geq 0\}$. Here the density function $\rho(x, y) = x^2 + y^2$.



$CM_x = \frac{1}{m(D)} \iint_D x\rho(x,y)\,dA$

$x\rho(x,y)$ is odd in x for a fixed y since x is odd & $\rho(x,y) = x^2+y^2$ is even & the region D is symmetric across the y-axis

therefore $CM_x = 0$ +1

$\rho(x,y) = x^2+y^2 = r^2$

$x^2+y^2 \leq 4$

$r^2 \leq 4$

$r \leq 2$

$0 \leq \theta \leq \pi$

or

$0 \leq y \leq \sqrt{4-x^2}$

$-2 \leq x \leq 2$

$M(D) = \int_0^{\pi}\int_0^2 r^2 r\,dr\,d\theta = \pi \cdot \frac{r^3}{3}\Big|_0^2 = \pi\left(\frac{2^3}{3} - 0\right) = \frac{8}{3}\pi$ +0.5

$CM_y = \frac{1}{m(D)}\int_{-2}^{2}\int_0^{\sqrt{4-x^2}} y(x^2+y^2)\,dy\,dx = \frac{3}{8\pi}\int_{-2}^{2}\left(\frac{x^2y^2}{2} + \frac{y^4}{4}\right)\Big|_0^{\sqrt{4-x^2}} dx$ +0.5

$16 - 8x^2 + x^4$

$= \frac{3}{8\pi}\int_{-2}^{2}\left(\frac{x^2}{2}(4-x^2) + \frac{1}{4}(4-x^2)^2\right)dx$

$= \frac{3}{8\pi}\int_{-2}^{2}\left(2x^2 - \frac{1}{2}x^4 + 4 - 2x^2 + \frac{1}{4}x^4\right)dx = \frac{3}{8\pi}\int_{-2}^{2}(4)\,dx = \frac{3}{8\pi}4x\Big|_{-2}^{2}$

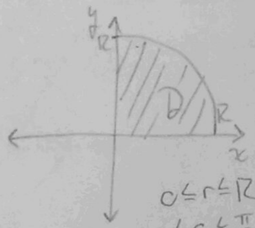
$= \frac{3}{8\pi}(8 - -8) = \frac{6}{\pi}$

$CM = \left(0, \frac{6}{\pi}\right)$

Problem 3. (4)

Find the iterated integral $\int_0^R \int_0^{\sqrt{R^2-x^2}} sin^2((x^2+y^2))dydx$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.



$0 \le r \le R$

$0 \le \theta \le \frac{\pi}{2}$

$\sin^2((x^2+y^2)) = \sin^2(r^2)$

$$\int_0^R \int_0^{\sqrt{R^2-x^2}} \sin^2((x^2+y^2))dydx = \int_0^{\frac{\pi}{2}} \int_0^R \sin^2(r^2)\, r\,dr\,d\theta$$

Let $u = r^2$, $du = 2r\,dr$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \int_0^{R^2} \sin^2 u\, du\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{u}{2} - \frac{\sin(2u)}{4}\Big|_0^{R^2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{R^2}{2} - \frac{\sin(2R^2)}{4} - (0-0)\right) d\theta = \frac{R^2}{4} - \frac{\sin(2R^2)}{8}\int_0^{\frac{\pi}{2}} d\theta$$

$$= \left(\frac{R^2}{4} - \frac{\sin(2R^2)}{8}\right)\frac{\pi}{2}$$

$\int \cos^2\theta = I_1$ $\qquad I_1 + I_2 = \theta$.

$\int \sin^2\theta = I_2$ $\qquad I_1 - I_2 = \frac{\sin(2\theta)}{2}$

$$2I_2 = \theta - \frac{\sin(2\theta)}{2}$$

$$I_2 = \frac{\theta}{2} - \frac{\sin(2\theta)}{4}$$

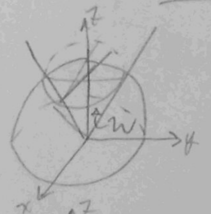
Problem 4. (4)

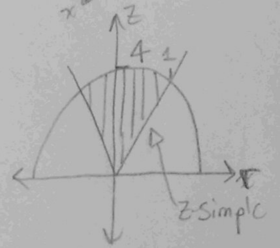
Find

$$\iiint_E x^2 + y^2 \, dxdydz,$$

where E is the finite solid bounded by the sphere $x^2 + y^2 + z^2 = 2z$ and the cone $z = \sqrt{x^2 + y^2}$

Note : E is inside both the sphere and the cone.





sphere $x^2+y^2+z^2=2z$

$r^2+z^2=2z$

$z^2-2z=-r^2$

$z^2-2z+4=4-r^2$

$(z-2)^2=4-r^2$

$z=2\pm\sqrt{4-r^2}$

cone $z=\sqrt{x^2+y^2}$

$z=\sqrt{r^2}=r$

intersection @ $x^2+y^2+(\sqrt{x^2+y^2})^2=2\sqrt{x^2+y^2}$

$2x^2+2y^2=2\sqrt{x^2+y^2}$

$2r^2=2r$

$2r^2-2r=0$

$r^2-r+\frac{1}{4}=\frac{1}{4}$

$(r-\frac{1}{2})^2=\frac{1}{4}$

$r-\frac{1}{2}=\pm\frac{1}{2}$

$r=\frac{1}{2}\pm\frac{1}{2}$

$0\le r\le 1$

$W=\left\{(r,\theta,z) \,\middle|\, \begin{matrix} r\le z\le 2+\sqrt{4-r^2} \\ 0\le r\le 1 \\ 0\le\theta\le 2\pi \end{matrix}\right\}$

$Vol(E)=\int_0^{2\pi}\int_0^1\int_r^{2+\sqrt{4-r^2}} r\,dz\,dr\,d\theta$

$=2\pi\int_0^1 (r\, z\big|_r^{2+\sqrt{4-r^2}})dr$

$=2\pi\int_0^1 r(2+\sqrt{4-r^2}-r)dr$

$=2\pi\int_0^1(2r+\sqrt{4-r^2}\,r-r^2)dr$

$=2\pi\left(\frac{2r^2}{2}\Big|_0^1-\frac{1\cdot 2}{2\cdot 3}(4-r^2)^{3/2}\Big|_0^1-\frac{r^3}{3}\Big|_0^1\right)$

$=2\pi\left(1^2-\frac{1}{3}(3\sqrt{3}-8)-\frac{1}{3}\right)=2\pi\left(1-\sqrt{3}+\frac{8}{3}-\frac{1}{3}\right)$

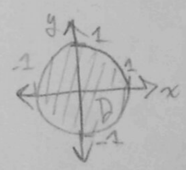
$=\cancel{\left(\frac{20}{3}-2\sqrt{3}\right)\pi}$

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{\cos(x^2+y^2)\cdot \sin z}\,dV,$$

where E is the region inside the cylinder $x^2+y^2=1$ with $0 \le z \le 1$.



$(0 \le r^2 \le 1)$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$0 \le z \le 1$

$\cos(2) \le \cos(x^2+y^2) \le \cos 0$; $\sin(0) \le z \le \sin(1)$

$\cos(2) \le \cos(x^2+y^2)\sin z \le \sin(1)$

$\iiint_E e^{\cos(2)}\,dV \le \iiint_E e^{\cos(x^2+y^2)\sin z}\,dV \le \iiint_E e^{\sin(1)}\,dV$

$\hookrightarrow \int_0^{2\pi}\int_0^1\int_0^1 e^{\cos 2}\, r\,dz\,dr\,d\theta = e^{\cos 2}(2\pi)\left(\frac{r^2}{2}\Big|_0^1\right)$

$= \pi e^{\cos 2}$

$\hookrightarrow \int_0^{2\pi}\int_0^1\int_0^1 e^{\sin 1}\, r\,dz\,dr\,d\theta = e^{\sin 1}(2\pi)\left(\frac{r^2}{2}\Big|_0^1\right)$

$= \pi e^{\sin 1}$

$\therefore \pi e^{\cos 2} \le \iiint_E e^{\cos(x^2+y^2)\sin z}\,dV \le \pi e^{\sin 1}$