1	4
2	4
3	3.5
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MATH 32B Midterm I, Spring 2015

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

Find the triple integral

$$\int\int\int_W \sqrt{x^2+z^2}\,dV,$$

where W is the region bounded by the paraboloid $4y = (x^2+z^2)$ and the plane $y = 4$.

Let $x^2+z^2=r^2$

$y = \frac{x^2+z^2}{4} \geq 0.$ $\quad y = \frac{r^2}{4} \leq 4.$

$0 \leq r \leq 4 \quad 0 \leq \theta \leq 2\pi$

$\frac{r^2}{4} \leq y \leq 4$

$$\int_0^{2\pi}\int_0^4\int_{\frac{r^2}{4}}^4 r^2 \, dy\,dr\,d\theta$$

$$= 2\pi \int_0^4 4r^2 - \frac{r^4}{4}\, dr$$

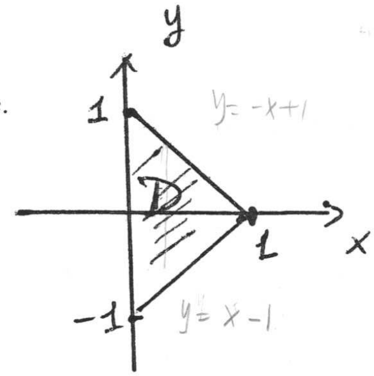
$$= 2\pi \left(\frac{4r^3}{3} - \frac{r^5}{20} \right)\Big|_0^4$$

$$= \frac{512\pi}{3} - \frac{512\pi}{5}$$

$$= \frac{1024\pi}{15}$$

Problem 2. (4)

Compute the center of mass of D. Here the density fuction $\rho(x, y) = x$.



$$m(D) = \iint_D \rho \, dA$$

$$= \int_0^1 \int_{x-1}^{1-x} x \, dy \, dx$$

$$= \int_0^1 x(1-x) - x(x-1) \, dx$$

$$= x^2 - \frac{2}{3}x^3 \Big|_0^1 = \frac{1}{3}$$

$$m_y = \iint_D x\rho \, dA = \int_0^1 \int_{x-1}^{1-x} x^2 \, dy \, dx$$

$$= \int_0^1 x^2 (2-2x) \, dx$$

$$= \frac{2x^3}{3} - \frac{x^4}{2} \Big|_0^1$$

$$= \frac{1}{6} \qquad \bar{x} = \frac{1}{6} \Big/ \frac{1}{3} = \frac{1}{2}$$

$$m_x = \iint_D y\rho \, dA = \int_0^1 \int_{x-1}^{1-x} xy \, dy \, dx$$

$$= \frac{1}{2} \int_0^1 x\left[(1-x)^2 - (x-1)^2\right] dx$$

$$= 0 \qquad \bar{y} = 0$$

The center of mass is $(\frac{1}{2}, 0)$

k^2

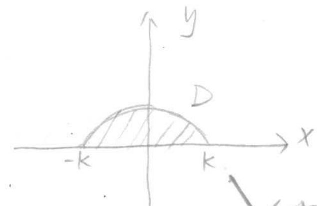
Problem 3. (4)

Find the iterated integral

$$\int_0^k \int_{y=-\sqrt{k^2-x^2}}^{\sqrt{k^2-x^2}} (x^2+y^2)\cos(x^4+y^4+2x^2y^2)\,dy\,dx.$$

$(x^2+y^2)^2$

Here k is a positive constant.



The area is bounded by $x \geq 0$, $x^2+y^2=k^2$.

Let $x^2+y^2=r^2$ $\quad 0 \le r \le k \quad 0 \le \theta \le \pi$

$$\int_0^{\pi}\int_0^{k} r^3 \cdot \cos(r^4)\,dr\,d\theta$$

Let $r^4=u$, $\quad du = 4r^3dr$

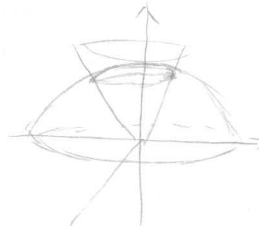
$$= \int_0^{\pi} \frac{\sin(r^4)}{4}\Big|_0^k \,d\theta$$

$$\int r^3\cos r^4 dr = \int \frac{\cos u}{4}du$$

$$= \frac{\sin u}{4}$$

$$= \frac{\pi \sin(k^4)}{4}$$

-0.5

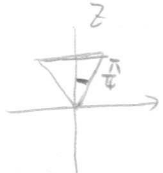


Problem 4. (4)

Find

$$\iiint_W \sqrt{x^2+y^2}\,dV.$$

Here W is the solid with $z \geq 0$, above the cone $x^2+y^2=z^2$ and below the sphere $x^2+y^2+z^2=R^2$. Here R is a positive constant.



At $y=0$
$x^2=z^2$
$\varphi=\frac{\pi}{4}$

$0 \leq \rho \leq R \qquad 0 \leq \varphi \leq \frac{\pi}{4} \qquad 0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^R \cancel{\rho^2 \sin^2\varphi} \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

[redacted]

Let $u=\cos\varphi$
$du=\sin\varphi\, d\varphi$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin^3\varphi \, d\varphi \int_0^R \rho^4 d\rho$$

$\int \sin^3\varphi \, d\varphi$

$= \int \sin^2\varphi \cdot \sin\varphi \, d\varphi$

$= \int (1-\cos^2\varphi)\sin\varphi\, d\varphi$

$= \int 1-u^2 \, du$

$= u - \frac{u^3}{3}$

$= \cos\varphi - \frac{\cos^3\varphi}{3}$

$$= 2\pi \cdot \left(\cos\varphi - \frac{\cos^3\varphi}{3}\right)\Big|_0^{\frac{\pi}{4}} \cdot \frac{R^5}{5}$$

$$= \frac{2\pi R^5}{5} \cdot \left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{12} - 1 + \frac{1}{3}\right)$$

$$= \frac{2\pi R^5}{5}\left(\frac{5\sqrt{2}}{12} - \frac{2}{3}\right)$$

$$= \frac{\sqrt{2}\pi R^5}{6} - \frac{4\pi R^5}{15}$$

X

$\frac{\pi}{4}$

$$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^R \rho^3 \sin^2\varphi \, d\rho\, d\varphi\, d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{4}} \sin^2\varphi \, d\varphi \int_0^R \rho^3 \, d\rho$$

$$= 2\pi \cdot \int_0^{\frac{\pi}{4}} \frac{1-\cos 2\varphi}{2} \Big|_0^{\frac{\pi}{4}} \cdot \frac{R^4}{4}$$

$$= \frac{\pi R^4}{2} \cdot \left[\frac{\pi}{8} - \frac{1}{2}\right]$$

$$= \frac{\pi^2 R^4}{16} - \frac{\pi R^4}{4}$$

Problem 5. (4)

Estimate the following integral

$$\iiint_W e^{\sin x \cos y \sin z} dV,$$

where W is the solid bounded by the cylinder $x^2 + y^2 = R^2$, the plane $z = -h$ and the plane $z = h$. Here R and h are two positive constants.

$$-1 \leq \sin x \cos y \sin z \leq 1$$

$$\Downarrow$$

$$\frac{1}{e} \leq e^{\sin x \cos y \sin z} \leq e$$

Therefore the integral should be between $\frac{1}{e} \cdot \text{Volume}(W)$ and $e \text{Volume}(W)$

$$\text{Vol}(W) = \iiint_W 1 \, dV$$

$$= \int_0^{2\pi} \int_0^R \int_{-h}^h r \, dz \, dr \, d\theta$$

$$= 2\pi \int_0^R r \cdot 2h \, dr$$

$$= 2\pi R^2 h$$

$$\frac{2\pi R^2 h}{e} \leq \iiint_W e^{\sin x \cos y \sin z} dV \leq e \cdot 2\pi R^2 h$$