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MATH 32B Midterm I, Fall 2018

Name: [redacted]

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Justify All Your Answers. No Points Will Be Given Without Sufficient Reasoning/Calculations.

Problem 1. (4)

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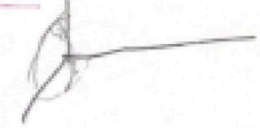
Find the triple integral $\iiint_W (y^2+z^2)\,dx\,dy\,dz$. Here W is the finite solid bounded by the surface $x = 1 - y^2 - z^2$ and the plane $x = 0$.

$\iiint_W r^2 \cdot r \, dx\,dr\,d\theta$

$y^2+z^2=r^2$



$\int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} r^3 \, dx\,dr\,d\theta$

$= \int_0^{2\pi}\int_0^{1} r^3 - r^5 \, dr\,d\theta$ ✓

$0 \le x \le 1-r^2$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$= \int_0^{2\pi} \frac{r^4}{4} - \frac{r^6}{6} \, d\theta \Big|_{r=0}^{1} = \left(\frac{1}{4}-\frac{1}{6}\right)-(0-0)$

$\left(\frac{1}{12}\right)\theta \Big|_{\theta=0}^{2\pi}$

$\frac{1}{4}-\frac{1}{6}$

$\frac{3}{12}-\frac{2}{12} = \frac{1}{12}$

$= \frac{\pi}{6}$ ✓

Problem 2. (4)

Compute the center of mass of D, where D is the finite region in $\mathbb{R}^2$ with $y \geq 0$ and bounded by $y = x$, $y = -x$, $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$. Here the density function $\rho(x, y) = y^2$.

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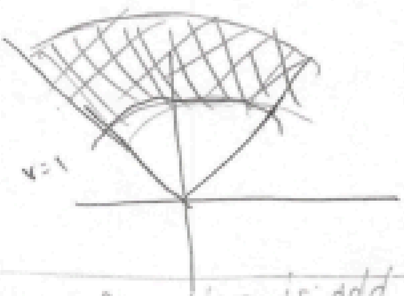
$r\sin\theta = r\cos\theta$

$r\sin\theta = -r\cos\theta$

$\pi/4 \leq \theta \leq 3\pi/4$

$1 \leq r \leq 2$



$x_{cm} = 0$ by symmetry because function is odd for x and symmetrical about y-axis.

$$\iint_D y^2\,dx\,dy = \int_{\pi/4}^{3\pi/4}\int_1^2 r^3\sin^2\theta\,dr\,d\theta = \int_{\pi/4}^{3\pi/4} \frac{r^4}{4}\sin^2\theta\,d\theta \Big|_{r=1}^{2}$$

$(4-\frac{1}{4})\sin^2\theta$

$$\frac{15}{4}\int_{\pi/4}^{3\pi/4} 1-\cos^2\theta\,d\theta \rightarrow \frac{15}{8}\int_{\pi/4}^{3\pi/4} 1+(1-2\cos^2\theta)\,d\theta =$$

$\div 2$ $\times 2$

$$\frac{15}{8}\int_{\pi/4}^{3\pi/4} 1-\cos(2\theta)\,d\theta = \frac{15}{8}\left(\theta - \frac{1}{2}\sin(2\theta)\right)\Big|_{\theta=\pi/4}^{3\pi/4}$$

$$\frac{15}{8}\left[\left(\frac{3\pi}{4}+\frac{1}{2}\right)-\left(\frac{\pi}{4}-\frac{1}{2}\right)\right] = \frac{15}{8}\left(\frac{\pi}{2}+1\right) = \text{Mass}$$

$$y_c = \iint_W y^3\,dx\,dy = \int_{\pi/4}^{3\pi/4}\int_1^2 r^4 \sin^3\theta\,dr\,d\theta = \int_{\pi/4}^{3\pi/4} \frac{r^5}{5}\sin^3\theta\,d\theta \Big|_{r=1}^{2}$$

$$\to \frac{32}{5} - \frac{1}{5}$$

$$\frac{31}{5}\int_{\pi/4}^{3\pi/4} \sin\theta(1-\cos^2\theta)\,d\theta = \frac{31}{5}\int_{\pi/4}^{3\pi/4} \sin\theta - \sin\theta\cos^2\theta\,d\theta =$$

$$\frac{31}{5}\left(-\cos\theta + \frac{1}{3}\cos^3\theta\right)\Big|_{\theta=\pi/4}^{3\pi/4} = \frac{31}{5}\left(\left(\frac{1}{\sqrt{2}} - \frac{1}{6\sqrt{2}}\right) - \left(\frac{-1}{\sqrt{2}} + \frac{1}{6\sqrt{2}}\right)\right)$$

$$= \frac{31}{5}\left(\frac{2}{\sqrt{2}}\cdot\frac{6}{6} - \frac{2}{6\sqrt{2}}\right) = \frac{31}{5}\left(\frac{10}{6\sqrt{2}}\right) = \frac{31}{\cancel{5}}\left(\frac{\cancel{5}}{3\sqrt{2}}\right) \quad \frac{31}{3\sqrt{2}} = y_c$$

$$y_{cm} = \frac{\frac{31}{3\sqrt{2}}}{\frac{15}{8}(\pi/2+1)} = \boxed{\frac{248}{45\sqrt{2}(\pi/2+1)}}$$

$$\begin{array}{r} 31 \\ \times 8 \\ \hline 248 \end{array} \qquad \begin{array}{r} 62 \\ \times 8 \\ \hline 496 \end{array}$$

$$\boxed{\text{center of mass: } \left(0, \frac{248}{45\sqrt{2}(\pi/2+1)}\right)}$$

$$\begin{array}{r} 31 \\ 8 \\ \hline 248 \end{array}$$

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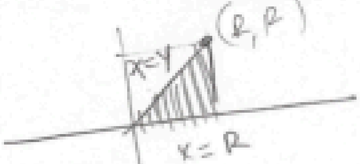
Problem 3. (4)

Find the iterated integral $\int_0^R \int_{x=y}^R y\cos(x^3)\,dx\,dy$. Here R is a positive constant.

Hint: Convert the iterated integral into a double integral and evaluate the double integral.



$$\frac{y^2}{2}\Big|_0^x \to \frac{x^2}{2} - 0$$

$$\int_0^R\int_0^x y\cos(x^3)\,dy\,dx = \int_0^R \frac{x^2}{2}\cos(x^3)\,dx =$$

$$\frac{1}{6}\sin(x^3)\Big|_0^R = \boxed{\frac{1}{6}\sin(R^3)}$$

✓

Problem 4. (4)

Find

$$\iiint_E x^2 + y^2 \, dxdydz,$$

where E is the finite solid bounded by the sphere $x^2 + y^2 + z^2 = R^2$ and the cone $x^2 + y^2 - z^2 = 0$. Here R is a positive constant.
Note : E is inside both the sphere and the cone.

Ice Cream cone.

$x^2+y^2=z^2 \quad \phi = \pi/4$

$\cancel{\rho^2}\sin^2\phi = \cancel{\rho^2}\cos^2\phi$

$\tan\phi = 1$

$0 \le \rho \le R$

$0 \le \phi \le \pi/4$

$0 \le \theta \le 2\pi$

2 $\int_0^{2\pi}\int_0^{\pi/4}\int_0^{R} \rho^2\sin^2\phi \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

Independent

$\int_0^{2\pi} d\theta \cdot \int_0^R \rho^4 d\rho \cdot \int_0^{\pi/4} \sin\phi(1-\cos^2\phi)\, d\phi =$

$\frac{2\pi R^5}{5} \int_0^{\pi/4} \sin\phi - \sin\phi\cos^2\phi \, d\phi = \frac{2\pi R^5}{5}\left(-\cos\phi + \frac{1}{3}\cos^3\phi\right)\Big|_{\phi=0}^{\pi/4}$

$= \frac{2\pi R^5}{5}\left(\left[\frac{-1}{\sqrt{2}} + \frac{1}{6\sqrt{2}}\right] - \left[-1 + \frac{1}{3}\right]\right) =$ -1

$\frac{2\pi R^5}{5}\left(\frac{-5}{6\sqrt{2}} + \frac{2}{3}\right) = \boxed{\frac{2\pi R^5}{5}\left(\frac{4\sqrt{2}-5}{6\sqrt{2}}\right)}$

Problem 5. (4)

Estimate the following integral

$$\iiint_E e^{(x^2+y^2)\cdot \sin z}\, dV,$$

where E is the solid inside the cylinder $x^2+y^2=R^2$ with $-R \le z \le R$. Here R is a positive constant.

$-1 \le \sin z \le 1$

$\iiint_E e^{-(x^2+y^2)} \le \iiint_E e^{(x^2+y^2)\sin z} \le \iiint_E e^{(x^2+y^2)}$

$\int_0^{2\pi}\int_0^R\int_{-R}^R e^{-r^2}\cdot r\, dz\,dr\,d\theta = 2R\int_0^{2\pi}\int_0^R re^{-r^2}\,dr\,d\theta =$

$2R\int_0^{2\pi}\int_0^R re^{-r^2}\,dr\,d\theta = \cancel{2}R\int_0^{2\pi} -\frac{1}{\cancel{2}} e^{-r^2}\Big|_0^R d\theta = R\int_0^{2\pi} 1-e^{-R^2} = 2\pi R(1-e^{-R^2})$

$\int_0^{2\pi}\int_0^R\int_{-R}^R re^{r^2}\,dz\,dr\,d\theta = 2R\int_0^{2\pi}\int_0^R re^{r^2}\,dr\,d\theta = \cancel{2}R\int_0^{2\pi}\frac{1}{\cancel{2}}e^{r^2}\,d\theta\Big|_{r=0}^{R}$

$R\int_0^{2\pi} e^{R^2}-1\,d\theta = 2\pi R(e^{R^2}-1)$

$$\boxed{2\pi R(1-e^{-R^2}) \le \iiint_E e^{(x^2+y^2)\sin z}\,dV \le 2\pi R(e^{R^2}-1)}$$

Estimation between these two values.

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