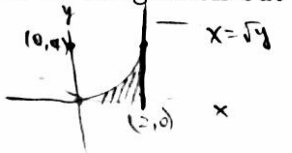
1. (10 points)

a. Change the order of integration but do not evaluate the integral $\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\, dx\, dy$.



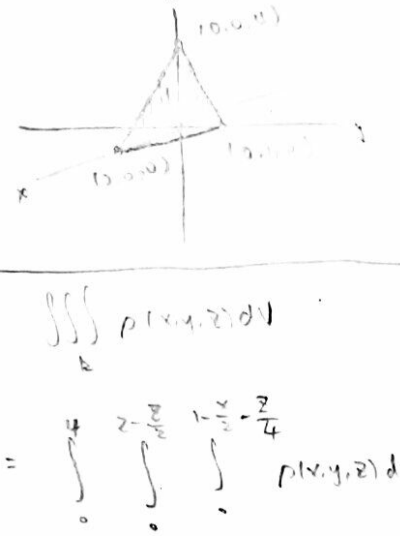
$x = \sqrt{y}$

x $\qquad$ $y = x^2$

$$\boxed{\int_0^{2}\int_0^{x^2} \sqrt{x^3+1}\, dy\, dx}$$

b. Write as an iterated integral with respect to the volume element $dy\, dx\, dz$ the integral $\iiint_R \rho(x,y,z)\, dV$ where R is the region in the first octant ($x,\ y,\ z \geq 0$) that lies under the plane $2x + 4y + z = 4$.

$\frac{x}{2} + \frac{y}{1} + \frac{z}{4} = 1$



$$\boxed{\iiint_R \rho(x,y,z)\, dV = \int_0^4 \int_0^{2-\frac{z}{2}} \int_0^{1-\frac{x}{2}-\frac{z}{4}} \rho(x,y,z)\, dy\, dx\, dz}$$

$2x + 4y + z = 4$

$4y = 4 - 2x - z$

$y = 1 - \frac{1}{2}x - \frac{1}{4}z$

at $y = 0$,

$2x + z = 4$

$2x = 4 - z$

$x = 2 - \frac{z}{2}$

2. (10 points)

a. Evaluate the integral $\displaystyle\int_0^1 \int_x^1 (y+x)\,dy\,dx$.

$$= \int_0^1 \frac{y^2}{2} + xy \Big|_x^1 \, dx$$

$$= \int_0^1 \left(\frac{1}{2} + x\right) - \left(\frac{x^2}{2} + x^2\right) dx$$

$$= \int_0^1 \frac{1}{2} - \frac{3x^2}{2} + x \, dx$$

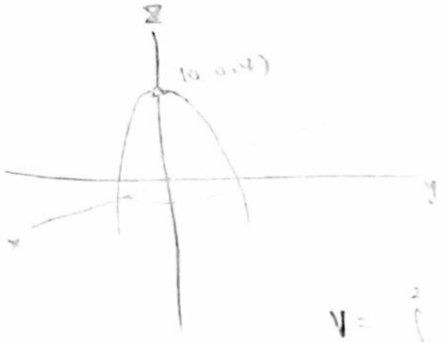
$$= \frac{1}{2}x - \frac{3x^3}{6} + \frac{x^2}{2} \Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{2}$$

$$= \frac{1}{2} \checkmark$$

5

b. Set up as a triple integral and then calculate the volume of the region that lies under the paraboloid $z = 4 - x^2 - y^2$ and above the plane $z = 0$.



$x = r\cos\theta$

$y = r\sin\theta$

$z = 4 - r^2$ ✓

when $z = 0$,

$r = \cancel{\pm}2$

↑ cannot be negative

$$V = \int_0^2 \int_0^{2\pi} \int_0^{4-r^2} r \, dz \, d\theta \, dr$$

$$= \int_0^2 \int_0^{2\pi} rz \Big|_0^{4-r^2} d\theta \, dr$$

5

$$= \int_0^2 \int_0^{2\pi} 4r - r^3 \, d\theta \, dr$$

$$= 2\pi \int_0^2 4r - r^3 \, dr$$

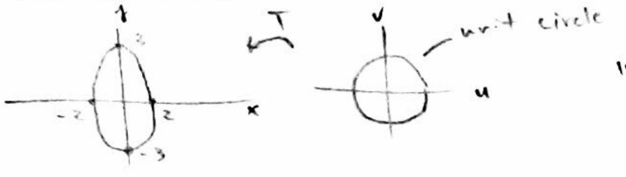
$$= 2\pi \left(\frac{4r^2}{2} - \frac{r^4}{4} \Big|_0^2 \right)$$

$$= 2\pi \left(\frac{16}{2} - \frac{16}{4} \right)$$

$$= 8\pi \checkmark$$

3. **(15 points)**

a. Use the appropriate change of coordinates to calculate the area bounded by the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. You must show your work to obtain credit.



let $u = \frac{x}{2}$, $v = \frac{y}{3}$,

so $u^2 + v^2 = 1$

$x = 2u$, $y = 3v$

$\text{Jac}(T) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $\quad \det(\text{Jac}(T)) = 6$

$\text{Area} = \iint_{\text{unit circle}} 6 \, dA$

$= 6\pi$

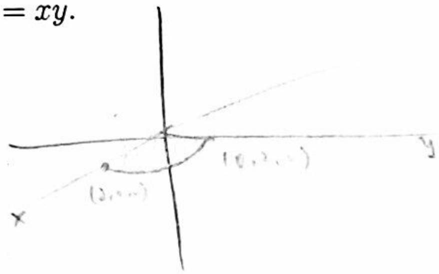
OR

$\text{Area} = \int_0^{2\pi}\int_0^1 6r \, dr \, d\theta$

$= 6\pi$

b. Set up **in polar coordinates** but do not evaluate an iterated integral that gives the x coordinate of the center of mass of a lamina in the plane given by $x^2 + y^2 \leq 4$ with $x, y \geq 0$ and with density function $\rho = xy$.

$4 - x^2 - y^2 \geq 0$



$x = r\cos\theta$

$0 \leq r \leq 2$ $\qquad \rho = r\cos\theta \, r\sin\theta$

$0 \leq \theta \leq \frac{\pi}{2}$ $\qquad = r^2\cos\theta\sin\theta$

$x\rho = r^3\cos^2\theta\sin\theta$

x-coord center of mass

$$\bar{x} = \frac{\int_0^{\pi/2}\int_0^2 x\rho \, r \, dr \, d\theta}{\int_0^{\pi/2}\int_0^2 \rho \, r \, dr \, d\theta}$$

$$\boxed{\bar{x} = \frac{\int_0^{\pi/2}\int_0^2 r^4\cos^2\theta\sin\theta \, dr \, d\theta}{\int_0^{\pi/2}\int_0^2 r^3\cos\theta\sin\theta \, dr \, d\theta}}$$

c. Let D be the the image of $R = [0,1] \times [0,1]$ under the map $G(u, v) = (3u + v, u - 2v)$. Calculate the integral $\iint_{D} x dx\, dy$ by converting it to an integral with respect to $du\, dv$.

$x = 3u + v$

$y = u - 2v$

$\text{Jac}(G) = \begin{bmatrix} 3 & 1 \\ 1 & -2 \end{bmatrix}$ $\quad$ $\det(\text{Jac}(G)) = -7$

$\iint_R x\, dx\, dy$

$= \iint_D (3u+v)(-7)\, du\, dv$

$= -7 \int_0^1 \int_0^1 3u + v \; du\, dv$

$= -7 \int_0^1 \frac{3}{2}u^2 + uv \Big|_0^1 dv$

$= -7 \int_0^1 \frac{3}{2} + v \; dv$

$= -7 \left(\frac{3}{2}v + \frac{v^2}{2} \right) \Big|_0^1$

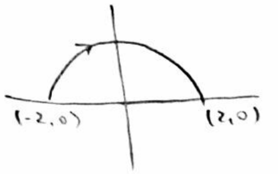
$= -7 \left(\frac{3}{2} + \frac{1}{2} \right)$

$= -7 \left(\frac{4}{2} \right)$

$= 14$

4. (15 points)

a. Find a parameterization for the curve that traces the **top** half of the circle $x^2 + y^2 = 4$, that starts at the point $(-2, 0)$ and goest to the point $(2, 0)$.



$\gamma(t) = (-2\cos t, 2\sin t)$, $0 \le t \le \pi$

5.

b. Suppose that $\gamma(t) = (\sin(t), \cos(t), t)$ with $0 \le t \le \frac{\pi}{2}$ parameterizes a curve Γ. Suppose that $\rho(x, y, z) = z^2$. Calculate the line integral $\int_\Gamma \rho \, ds$.

$\gamma'(t) = (\cos t, -\sin t, 1)$

$\|\gamma'(t)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

$\rho(\gamma(t)) = t^2$

5.

$$\int_\Gamma \rho \, ds$$

$$= \int_a^b \rho(\gamma(t)) \, \|\gamma'(t)\| \, dt$$

$$\int_0^{\frac{\pi}{2}} \sqrt{2} t^2 \, dt$$

$$= \sqrt{2} \; \frac{t^3}{3} \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\sqrt{2}}{24} \pi^3$$

c. Suppose that $\gamma(t) = (\sin(t), \cos(t))$ with $0 \leq t \leq \frac{\pi}{2}$ parameterizes a curve Γ. Suppose that $F(x, y, \cancel{z}) = \langle 2y, -2x \rangle$. Calculate the line integral $\int_\Gamma F \cdot d\vec{s}$.

$\gamma'(t) = (\cos t, -\sin t)$

$F(\gamma(t)) = \langle 2\cos t, -2\sin t \rangle$

$$\int_\Gamma \vec{F} \cdot d\vec{s}$$

$$= \int_a^b F(\gamma(t)) \cdot \gamma'(t)\, dt$$

$$= \int_0^{\pi/2} \langle 2\cos t, -2\sin t \rangle \cdot \langle \cos t, -\sin t \rangle\, dt$$

$$= \int_0^{\pi/2} 2\cos^2 t + 2\sin^2 t\, dt$$

$$= \int_0^{\pi/2} 2\, dt$$

$$= 2t \Big|_0^{\frac{\pi}{2}}$$

$$= \pi$$

5.