1. (a) $\boxed{\text{5 points}}$ Sketch the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 8 - x^2 - y^2$. Then set up a **double integral** which gives the volume of this region. (Do not evaluate the double integral).

(b) $\boxed{\text{5 points}}$ Compute the volume of the region bounded by $z = 16 - y$, $z = y$, $y = x^2$, and $y = 8 - x^2$

a) $x^2+y^2 = 8-x^2-y^2$

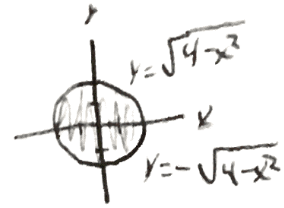
$2x^2+2y^2=8$

$x^2+y^2=4$

$y=\pm\sqrt{4-x^2}$

5

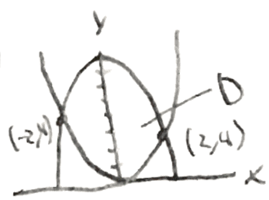
project onto xy-plane $\longrightarrow$



$z_1(x,y)=x^2+y^2 \quad z_2(x,y)=8-x^2-y^2$

$$\boxed{\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(8-x^2-y^2-(x^2+y^2)\right) dy\, dx}$$

b)



D: $-2 \le x \le 2$

$x^2 \le y \le 8-x^2$

$z_2(x,y)=16-y$

$z_1(x,y)=y$

$x^2=8-x^2$

$2x^2=8$

$x=2$

5

$$Vol = \iint_D \left(z_2(x,y)-z_1(x,y)\right)dA$$

$$= \int_{x=-2}^{2}\int_{y=x^2}^{8-x^2} (16-2y)\,dy\,dx$$

$$= \int_{y=-2}^{2} \left[(16y-y^2)\Big|_{x^2}^{8-x^2}\right] dx$$

$$= \int_{-2}^{2}\left[y(16-y)\Big|_{x^2}^{8-x^2}\right]dx = \int_{-2}^{2}\left[(8-x^2)(8+x^2)-x^2(16-x^2)\right]dx = \int_{-2}^{2}\left[64-x^4-16x^2+x^4\right]dx$$

$$= \int_{-2}^{2}(64-16x^2)\,dx = 64x-\frac{16}{3}x^3\Big|_{-2}^{2} = 2\left(128-\frac{128}{3}\right) = 2\left(\frac{256}{3}\right) = \boxed{\frac{512}{3}}$$

4
16
×8
128

2. $\boxed{\text{5 points}}$ Prove the inequality $\displaystyle\iint_D \frac{2}{1+x^2+y^2}\,dA \leq 8\pi$, where D is the disk $x^2+y^2 \leq 4$.

$f(x,y) = \dfrac{2}{1+x^2+y^2}$ is at a maximum when $1+x^2+y^2$ is at a minimum

$1+x^2+y^2$ is at a min when $(x,y)=(0,0)$. Therefore the maximum

value of $f(x)$ is $f(0,0)=2$, and $(0,0) \in D$

$M =$ max of $f(x)$ on $D = 2$

$\iint_D f(x)\,dA \leq \text{Area}(D) \cdot M$

Area of $D \Rightarrow$ Area of a circle with radius $r=2$

$\text{Area}(D) = \pi r^2 = 4\pi$

Therefore,

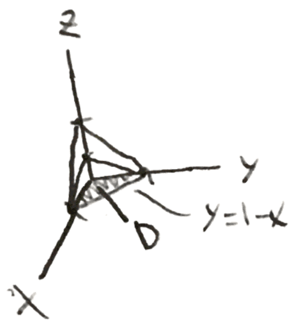
$\iint_D f(x)\,dA \leq 4\pi \cdot 2$

$$\boxed{\iint_D \frac{2}{1+x^2+y^2}\,dA \leq 8\pi}$$

✓

5

3. $\boxed{\text{5 points}}$ Using **triple integration** to compute the volume of the solid region $\mathcal{W}$ which lies in the first octant: $x \geq 0, y \geq 0, z \geq 0$ and bounded by the planes $x + y + z = 1$ and $x + y + 2z = 1$.



D: $0 \le x \le 1$
$0 \le y \le 1-x$

W: $0 \le x \le 1$
$0 \le y \le 1-x$
$\frac{1-x-y}{2} \le z \le 1-x-y$

$z_1 = \frac{1-x-y}{2}$

$z_2 = 1-x-y$

$$\text{Vol} = \iiint_W dV = \int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=\frac{1-x-y}{2}}^{1-x-y} dz\,dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1-x}\left[1-x-y-\frac{1-x-y}{2}\right] dy\,dx$$

$$= \frac{1}{2}\int_{x=0}^{1}\int_{y=0}^{1-x}(1-x-y)\,dy\,dx = \frac{1}{2}\int_0^1\left[(1-x)y-\frac{y^2}{2}\Big|_{y=0}^{1-x}\right]dx$$

$$= \frac{1}{2}\int_0^1\left[(1-x)^2-\frac{(1-x)^2}{2}\right]dx = \frac{1}{4}\int_0^1(1-x)^2\,dx$$

$$= -\frac{1}{12}(1-x)^3\Big|_0^1 = \boxed{\frac{1}{12}}$$

4. [5 points] Compute the following integral using **polar co-ordinates**. 5

$$\iint_D f(x,y),\ dA, \quad \text{where} \quad f(x,y) = (x^2+y^2)^{-\frac{3}{2}} \text{ and } D: x^2+y^2 \le 1, x+y \ge 1.$$

$r^2 \le 1$ $y \ge 1-x$

$$f(r\cos\theta, r\sin\theta) = (r^2)^{-3/2} = r^{-3}$$

$$\iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

$y = 1-x$

$r\sin\theta = 1 - r\cos\theta$

$r(\sin\theta + \cos\theta) = 1$

$r = \frac{1}{\sin\theta + \cos\theta}$

$$= \int_{\theta=0}^{\pi/2} \int_{r=(\sin\theta+\cos\theta)^{-1}}^{1} r^{-2}\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left[-\frac{1}{r} \Big|_{(\sin\theta+\cos\theta)^{-1}}^{1} \right] d\theta$$

$$= \int_{0}^{\pi/2} \left[\sin\theta + \cos\theta - 1 \right] d\theta$$

$$= \sin\theta - \cos\theta - \theta \Big|_0^{\pi/2} = 1 - \cancel{0} - \frac{\pi}{2} + (\cancel{0} + 1 - \cancel{0})$$

$$= \boxed{2 - \frac{\pi}{2}}$$