Math 32B-2 Yeliussizov. Midterm 2

Exam time: 6:00-7:30 PM, February 27, 2017

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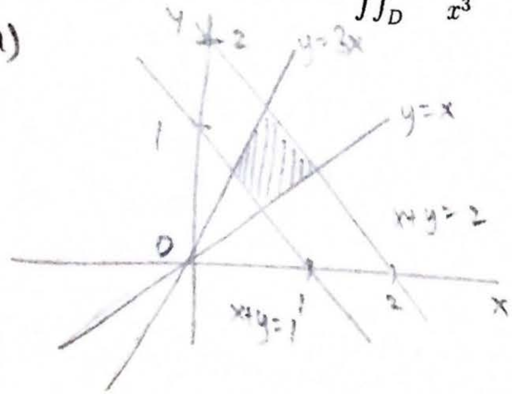
Discussion section: Tianqi 2A Tue, 2B Thu; Christian 2C Tue, 2D Thu; Kalyanswamy 2E Tue, 2F Thu

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (15)	P 2 (15)	P 3 (20)	P 4 (20)	P 5 (20)	Total (90 pt)
15	15	20	20	12	82

Problem 1. (15 points) Let D be the region enclosed by $y=x$, $y=3x$, $y=1-x$, $y=2-x$.
(a) (5 points) Find a map $F(x,y)$ whose image $F(D)$ is a rectangle (i.e., maps D to a rectangle)
(b) (10 points) Evaluate $\displaystyle\iint_D \frac{y(x+y)}{x^3}\,dxdy$ using change of variables from F.

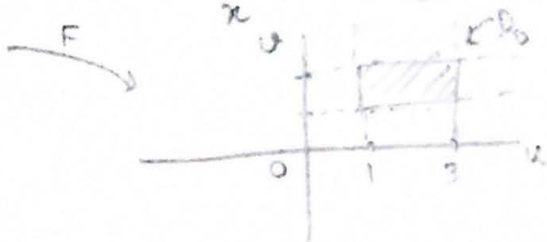
1. a)



$$F(x,y) = \left(\frac{y}{x},\ x+y\right)$$

Let $u=\frac{y}{x}$, $v=x+y$

$1\le u\le 3$
$1\le v\le 2$



b)

$$\iint_D \frac{y(x+y)}{x^3}\,dx\,dy$$

$$Jac(F)(x,y) = \det\begin{vmatrix} -y/x^2 & 1/x \\ 1 & 1 \end{vmatrix} = \left(-\frac{y}{x^2}-\frac{1}{x}\right) = -\left(\frac{x+y}{x^2}\right)$$

Over our domain D, $x,y\ge 0$

$$\Rightarrow |Jac(F)(x,y)| = \frac{(x+y)}{x^2}$$

Let $G(u,v) = (x(u,v), y(u,v))$. Using above change of variables

$$|Jac(G)(u,v)| = \frac{1}{|Jac(F)(x,y)|} = \frac{x^2}{x+y}$$

$$f(u,v) = \frac{y}{x^3}(x+y)\cdot|Jac(G)(u,v)| = \frac{y\cancel{(x+y)}}{\cancel{x^3}}\cdot\frac{\cancel{x^2}}{\cancel{(x+y)}} = \frac{y}{x} = u$$

$$\Rightarrow \iint_D \frac{y(x+y)}{x^3}\,dx\,dy = \int_1^3\int_1^2 u\,dv\,du = \int_1^3 u\,du\int_1^2 dv = \left(\frac{u^2}{2}\right)_1^3 (v)_1^2$$

$$= \left(\frac{9}{2}-\frac{1}{2}\right)(2-1) = (4)(1) = 4$$

Problem 2. (15 points) Let a, b, c be real constants. Show that $\operatorname{div}(\mathbf{F} \times \langle a, b, c\rangle) = 0$ if $\mathbf{F}$ is a conservative vector field.

Let $\vec{F} = \langle P, Q, R\rangle$ where P, Q, R are functions of x, y, z.

$$\vec{F} \times \langle a,b,c\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ P & Q & R \\ a & b & c \end{vmatrix} = \langle cQ - bR, -(cP - aR), bP - aQ\rangle$$

$$= \langle cQ - bR, aR - cP, bP - aQ\rangle$$

$$\operatorname{div}(\vec{F} \times \langle a,b,c\rangle) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \cdot \langle cQ - bR, aR - cP, bP - aQ\rangle$$

$$= c\frac{\partial Q}{\partial x} - b\frac{\partial R}{\partial x} + a\frac{\partial R}{\partial y} - c\frac{\partial P}{\partial y} + b\frac{\partial P}{\partial z} - a\frac{\partial Q}{\partial z}$$

$$= a\left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right) + b\left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right) + c\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$$

For a conservative vector field $\vec{F}$,

$$\operatorname{curl}(\vec{F}) = \vec{0} \Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle$$

$$\Rightarrow \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right\rangle = \langle 0, 0, 0\rangle$$

$$\Rightarrow \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0, \quad \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} = 0, \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

Plugging these back into original equation,

$$\operatorname{div}(\vec{F} \times \langle a,b,c\rangle) = a(0) + b(0) + c(0) = 0$$ (if $\vec{F}$ is a conservative vector field)

Problem 3. (20 points) Consider the path C parametrized by $\mathbf{r}(t) = (\cos 2t, \sin 2t, t)$ for $0 \le t \le 1$.
(a) (10 points) Evaluate the length of C.
(b) (10 points) Evaluate the vector line integral $\int_C \mathbf{F}\, d\mathbf{r}$, where $\mathbf{F} = \langle -y, x, z \rangle$.

3. a) Length of C = $\oint_C 1 . dr$

$\vec{r}(t) = \langle \cos 2t, \sin 2t, t \rangle$

$\vec{r}'(t) = \langle -2\sin 2t, 2\cos 2t, 1 \rangle$

$\|\vec{r}'(t)\| = \sqrt{4\sin^2 2t + 4\cos^2 2t + 1} = \sqrt{4(\sin^2 2t + \cos^2 2t) + 1} = \sqrt{4+1} = \sqrt{5}$

Length of C = $\int_0^1 1 . \sqrt{5}\, dt = \sqrt{5}\,(t)\Big|_0^1 = \sqrt{5}$

b) $\vec{F}(\vec{r}(t)) = \langle -\sin 2t, \cos 2t, t \rangle$

$\int_C \vec{F}\, d\vec{r} = \int_0^1 \langle -\sin 2t, \cos 2t, t \rangle \cdot \langle -2\sin 2t, 2\cos 2t, 1 \rangle\, dt$

$= \int_0^1 (2\sin^2 2t + 2\cos^2 2t + t)\, dt$

$= \int_0^1 (t+2)\, dt = \left(\frac{t^2}{2} + 2t\right)\Big|_0^1 = \left(\frac{1}{2} + 2\right) - 0 = \frac{5}{2}$

Problem 4. (20 points) Let C be a path from $(2,0)$ to $(0,1)$ along the ellipse $x^2+4y^2=4$ in the first quadrant, oriented counterclockwise.

(a) (10 points) Let $\mathbf{F} = \langle -y\sin x, x+\cos x\rangle$. Show that $\mathbf{F}$ is conservative, find a potential function $f(x,y)$ so that $\mathbf{F} = \nabla f$, and evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

(b) (10 points) Let $\mathbf{F} = \langle -y, x\rangle$. Is it conservative? Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$.

4. a) $\vec{F} = \langle y - y\sin x,\ x + \cos x\rangle$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y - y\sin x & x+\cos x & 0 \end{vmatrix} = \langle 0, 0, (1-\sin x) - (1-\sin x)\rangle = \langle 0,0,0\rangle$$

Since $\text{curl}(\vec{F}) = \vec{0}$ and $\vec{F}$ is defined everywhere, it is conservative.

$\vec{F} = \nabla f \quad \Rightarrow \quad \frac{\partial f}{\partial x} = y - y\sin x \quad \Rightarrow f = \int (y - y\sin x)\,dx$

$= xy + y\cos x + C$

$\frac{\partial f}{\partial y} = x + \cos x$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(xy + y\cos x + C) = x + \cos x + \frac{\partial C}{\partial y} = x + \cos x$$

$\Rightarrow \frac{\partial C}{\partial y} = 0, \quad \Rightarrow$ Let $C = 0$

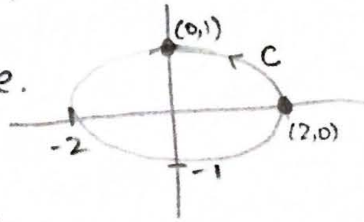
$\therefore f(x,y) = xy + y\cos x$

Since $\vec{F}$ is conservative, $\int_C \vec{F}\cdot d\vec{r} = f(0,1) - f(2,0)$

$= (0+1) - (0) = 1$

b) $\vec{F} = \langle -y, x\rangle \qquad \text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \langle 0, 0, 2\rangle$

Since $\text{curl}(\vec{F}) \neq 0$, $\vec{F}$ is not conservative.



Parametrization of curve: $\vec{r}(\theta) = \langle 2\cos\theta, \sin\theta\rangle$
$0 \le \theta \le \pi/2$

$\vec{r}\,'(\theta) = \langle -2\sin\theta, \cos\theta\rangle$

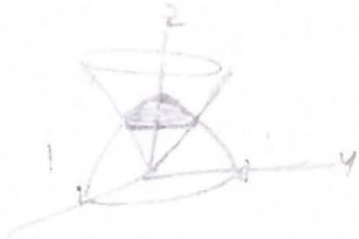
$\vec{F}(\vec{r}(\theta)) = \langle -\sin\theta, 2\cos\theta\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{\pi/2} \langle -\sin\theta, 2\cos\theta\rangle \cdot \langle -2\sin\theta, \cos\theta\rangle\, d\theta = \int_0^{\pi/2} (2\sin^2\theta + 2\cos^2\theta)\,d\theta = 2\int_0^{\pi/2} d\theta$$

$= 2\left(\frac{\pi}{2} - 0\right) = \pi$

Problem 5. (20 points) Compute the area of the surface enclosed by the cone $z = \sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2 = 1$ from above.

$$\text{Area} = \int_S \|\vec{N}(\theta,\phi)\|\, d\phi\, d\theta$$



$$\vec{r}(\theta,\phi) = (\sin\phi\cos\theta,\ \sin\phi\sin\theta,\ \cos\phi)$$

$$0 \le \theta \le 2\pi$$

$$\cos\phi = \sqrt{\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta}$$

$$\Rightarrow \cos\phi = \sin\phi \qquad \Rightarrow \quad 0 \le \phi \le \frac{\pi}{4}$$

$$\Rightarrow \quad \phi = \frac{\pi}{4}$$

$$\vec{T}_\theta = \langle -\sin\phi\sin\theta,\ \sin\phi\cos\theta,\ 0\rangle$$

$$\vec{T}_\phi = \langle \cos\phi\cos\theta,\ \cos\phi\sin\theta,\ -\sin\phi\rangle$$

$$\vec{N}(\theta,\phi) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \\ \cos\phi\cos\theta & \cos\phi\sin\theta & -\sin\phi \end{vmatrix} = \langle -\sin^2\phi\cos\theta,\ -\sin^2\phi\sin\theta,\ -\sin\phi\cos\phi\sin^2\theta - \sin\phi\cos\phi\cos^2\theta\rangle$$

$$= \langle -\sin^2\phi\cos\theta,\ -\sin^2\phi\sin\theta,\ -\sin\phi\cos\phi\rangle$$

$$\|\vec{N}(\theta,\phi)\| = \sqrt{(-\sin^2\phi\cos\theta)^2 + (-\sin^2\phi\sin\theta)^2 + (-\sin\phi\cos\phi)^2}$$

$$= \sqrt{\sin^4\phi(\sin^2\theta + \cos^2\theta) + \sin^2\phi\cos^2\phi} = \sqrt{\sin^2\phi(\sin^2\phi + \cos^2\phi)} = \sqrt{\sin^2\phi}$$

Since $0 \le \phi \le \pi/4$, $\sin\phi > 0$

$$\Rightarrow \|\vec{N}(\theta,\phi)\| = \sin\phi$$

$$\text{Surface Area} = \int_0^{2\pi}\int_0^{\pi/4} \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/4} \sin\phi\, d\phi = (2\pi - 0)\left(-\cos\phi\right)\Big|_0^{\pi/4}$$

$$= 2\pi\left(\cos 0 - \cos\frac{\pi}{4}\right) = 2\pi\left(1 - \frac{1}{\sqrt{2}}\right)$$

− cone?