b)

$z = 2x$ $\quad z = x^2+y^2$ $\quad 0 \le r \le 2\cos\theta$

$0 \le \theta \le 2\pi$

$G(x,y) = \langle x, y, 2x \rangle$

$G_x = \langle 1, 0, 2 \rangle$

$G_y = \langle 0, 1, 0 \rangle$

$G_x \times G_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 0 & 1 & 0 \end{vmatrix} = \langle -2, 0, 1 \rangle$

$\| \quad \| = \sqrt{2^2+1^2} = \boxed{\sqrt{5}}$

$$\int_0^{2\pi} \int_0^{2\cos\theta} r\cos\theta \sqrt{5}\; r\,dr\,d\theta$$

$$\sqrt{5} \int_0^{2\pi} \int_0^{2\cos\theta} r^2 \cos\theta$$

$$\frac{1}{4} \int_0^{\pi} \cos 2\theta^2$$

$$\frac{1}{4} \left(\int_0^{\pi} \frac{1}{2} + \frac{\cos 4\theta}{2} \right)$$

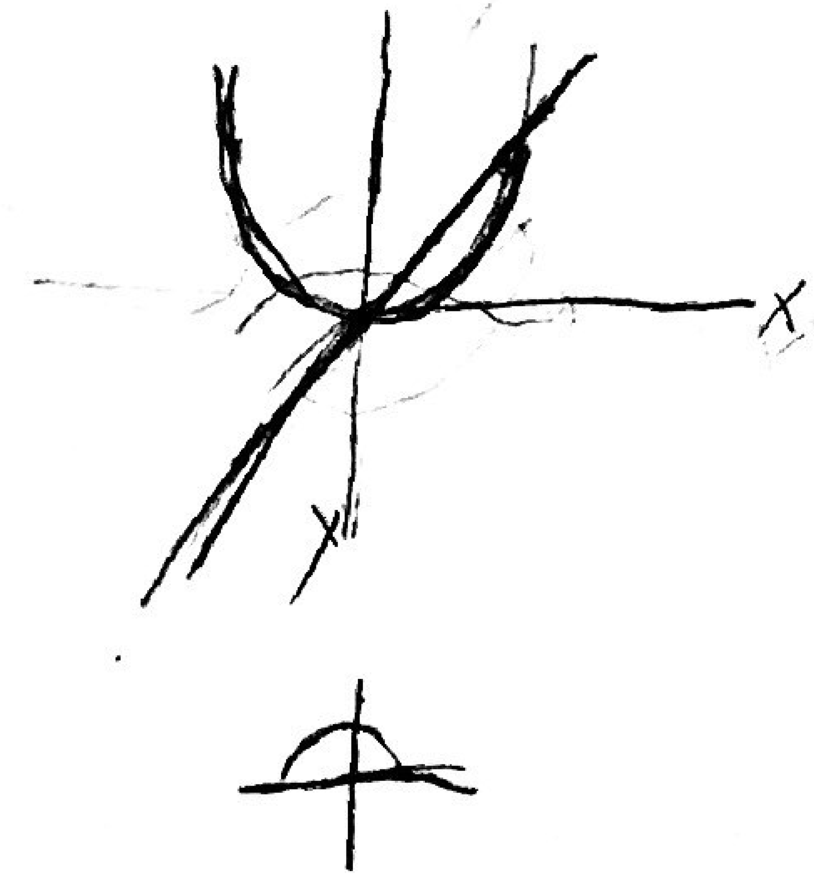
$$\frac{1}{4} \left[\frac{\theta}{2} + \frac{\sin 4\theta}{8} \right]_0^{\pi}$$

$$\frac{1}{4} \left(\frac{\pi}{2} + 0 \right)$$

$$\boxed{\frac{\pi}{8}}$$

Problem 5. (15 points) Let S be the part of the plane $z = 2x$ contained in the paraboloid $z = x^2 + y^2$.
(a) (8 points) Evaluate the area of S.
(b) (7 points) Evaluate $\iint_S x\, dS$

a)



$x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta$

$0 \le \theta \le \pi$

$0 \le r \le 2\cos\theta$

$$G(r,\theta) = \langle r\cos\theta,\ r\sin\theta,\ 2r\cos\theta \rangle$$

$$G_r = (\cos\theta, \sin\theta, 2\cos\theta)$$

$$G_\theta = (-r\sin\theta,\ r\cos\theta,\ -2r\sin\theta)$$

$$\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2\cos\theta \\ -r\sin\theta & r\cos\theta & -2r\sin\theta \end{vmatrix} = \langle -2r\sin^2\theta - 2r\cos^2\theta,\ 2r\sin\theta\cos\theta - 2r\sin\theta\cos\theta,\ r\cos^2\theta + r\sin^2\theta \rangle$$

$$= \boxed{\langle -2r, 0, r \rangle}$$

$$\|G_r \times G_\theta\| = \sqrt{4r^2 + r^2} = \boxed{\sqrt{5}\, r}$$

$$\int_0^\pi \int_0^{2\cos\theta} \sqrt{5}\, r\, dr\, d\theta \quad ① \ \sqrt{5}\int_0^{2\cos\theta} r\, dr = \sqrt{5}\left(\frac{r^2}{2}\Big|_0^{2\cos\theta}\right) = \sqrt{5}\left(\frac{4\cos^2\theta}{2}\right) = 2\sqrt{5}\cos^2\theta$$

$$② \ 2\sqrt{5}\int_0^\pi \cos^2\theta\, d\theta = 2\sqrt{5}\int_0^\pi \frac{1}{2} + \frac{\cos 2\theta}{2}\, d\theta = 2\sqrt{5}\left(\frac{1}{2}\theta + \frac{\sin 2\theta}{4}\Big|_0^\pi\right)$$

$$= 2\sqrt{5}\left(\frac{\pi}{2} + 0\right)$$

$$= \boxed{\pi\sqrt{5}}$$

$$\int_0^\pi \int_0^{2\cos\theta} r^2\cos\theta\sqrt{5}\, dr\, d\theta$$

$$① \ \sqrt{5}\int_0^{2\cos\theta} r^2\cos\theta\, dr = \sqrt{5}\left(\frac{r^3}{3}\cos\theta\Big|_0^{2\cos\theta}\right) = \sqrt{5}\left(\frac{8\cos^3\theta}{3}\cos\theta\right)$$

$$② \ \frac{8}{3}\sqrt{5}\int_0^\pi \cos^3\theta\cos\theta\, d\theta$$

$$= (\cos^2\theta)(\cos^2\theta)$$

$$\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)\left(\frac{1}{2} + \frac{\cos 2\theta}{2}\right)$$

$$\frac{1}{4} + \frac{\cos 2\theta}{4} + \frac{\cos 2\theta}{4} + \frac{\cos^2 2\theta}{4}$$

$$\frac{\cos^2 2\theta}{4} + \frac{\cos 2\theta}{2} + \frac{1}{4}$$

$$\frac{8}{3}\sqrt{5}\left(\int_0^\pi \frac{\cos^2 2\theta}{4}\, d\theta + \int_0^\pi \frac{\cos 2\theta}{2}\, d\theta + \int_0^\pi \frac{1}{4}\, d\theta\right)$$

$$\frac{8}{3}\sqrt{5}\left(\int_0^\pi \frac{\cos^2 2\theta}{4} + \left[\frac{\sin 2\theta}{4}\right]_0^\pi + \left[\frac{\theta}{4}\right]_0^\pi\right)$$

$$\frac{8}{3}\sqrt{5}\left(\int_0^\pi \frac{\cos^2 2\theta}{4} + 0 + \frac{\pi}{4}\right) \quad \text{back}$$

$$\frac{8}{3}\sqrt{5}\left(\frac{\pi}{8} + \frac{\pi}{4}\right) = \frac{\cancel{8}}{\cancel{3}}\sqrt{5}\left(\frac{\cancel{3}\pi}{\cancel{8}}\right) = \boxed{\pi\sqrt{5}}$$

$$\left(\frac{\pi}{8} + \frac{2\pi}{8}\right)$$

Problem 4. (10 points) Consider the vector field $\mathbf{F} = \langle \sin y, x\cos y + \cos z, -y\sin z + \cos z \rangle$. Evaluate $\int_C \mathbf{F}\, d\mathbf{r}$, where C is the path given by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$ for $0 \le t \le \pi/2$.

$F = \langle \sin y, x\cos y + \cos z, -y\sin z + \cos z \rangle$ Conserv? yes!

$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$, $\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$

$\cos y = \cos y$ ✓, $-\sin z = -\sin z$ ✓, $0 = 0$ ✓

$$\left.\begin{aligned} f_x &= \sin y \\ f_y &= x\cos y + \cos z \\ f_z &= -y\sin z + \cos z \end{aligned}\right\}$$

$\underline{x\sin y} + C(y,z)$

$x\cos y + C'(y) = x\cos y + \cos z$

$C'(y) = \cos z$

$C(y) = \underline{y\cos z}$

$x\sin y + y\cos z$ (partial w/ resp to z)

$-y\sin z + C'(z) = -y\sin z + \cos z$

$C'(z) = \cos z$

$C(z) = \underline{\sin z}$

pot. function

$\boxed{f(x,y,z) = x\sin y + y\cos z + \sin z}$

$r(\pi/2) = \langle \cos\pi/2, \sin\pi/2, \pi/2 \rangle$

$= \boxed{\langle 0, 1, \pi/2 \rangle}$ finish

$r(0) = \langle \cos 0, \sin 0, 0 \rangle$

$\boxed{\langle 1, 0, 0 \rangle}$ start

✓

$f(\text{finish}) - f(\text{start}) =$

$f(0, 1, \pi/2) = 0 + \cos\pi/2 + \sin\pi/2$

$= \boxed{1}$

$f(1,0,0) = \sin 0 + 0 + \sin 0$

$= \boxed{0}$

$1 - 0 = \boxed{1}$ ✓

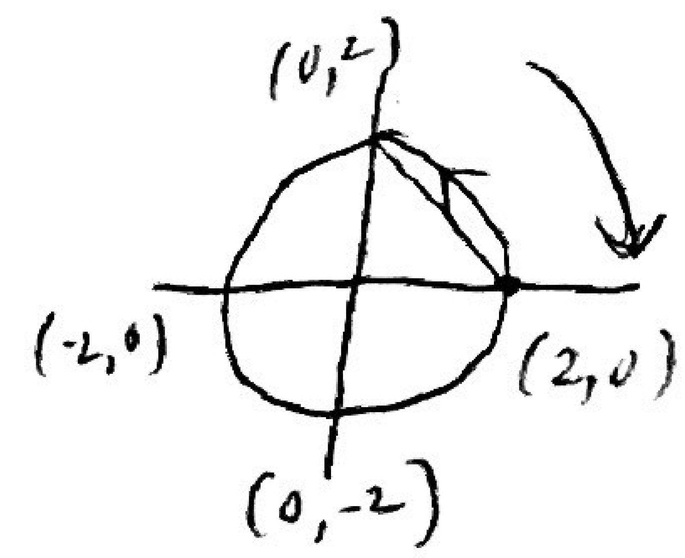
$r(t) = (0,2) + t(2,-2)$ line

closed $(2,0$ to $0,2)$

Problem 3. (15 points) Let C be the closed path from $(2,0)$ to $(0,2)$ along the quarter circle $x^2+y^2=4$ (counterclockwise), and then going back from $(0,2)$ to $(2,0)$ along the straight line segment.

(a) (7 points) Evaluate the scalar integral $\int_C y\, ds$

(b) (8 points) Evaluate the vector line integral $\int_C \langle 2y, x\rangle\, dr$



7

a)

$r_{circle}(t) = (2\cos t, 2\sin t)$ $\quad r_{line}(t) = (2t, -2t+2)$

$r'_{circle}(t) = (-2\sin t, 2\cos t)$ $\quad r'_{line}(t) = (2,-2)$

$0 \le t \le \pi/2$ $\quad\quad 0 \le t \le 1$

scalar $f(r(t)) \cdot \|r'(t)\|$

$\|r'_{circle}\| = \sqrt{4\sin^2 t + 4\cos t^2}$ $\quad \|r'_{line}\| = \sqrt{4+4}$

$\|r'_{circle}\| = \boxed{2}$ $\quad \|r'_{line}\| = \boxed{2\sqrt{2}}$

$\int_{circle} 2\sin t\,(2)\,dt + \int_{line} -2t+2\,(2\sqrt{2})\,dt$

$\int_0^{\pi/2} 4\sin t\,dt + \int_0^1 -4\sqrt{2}\,t + 4\sqrt{2}\,dt = -4\cos t\Big|_0^{\pi/2} - 2\sqrt{2}t^2\Big|_0^1 + 4\sqrt{2}t\Big|_0^1$

$= \boxed{(0+4) - (2\sqrt{2}) + (4\sqrt{2})}$

$\boxed{4+2\sqrt{2}}$ ✓

b) $F = \langle 2y, x\rangle$

Conserv?

$2 \ne 1$ No! ✓

$\int_{circle} \langle 4\sin t, 2\cos t\rangle \cdot \langle -2\sin t, 2\cos t\rangle dt + \int_{line} \langle -4t+4, 2t\rangle \cdot \langle 2,-2\rangle\, dt$

$\int_0^{\pi/2} -8\sin^2 t + 4\cos^2 t\, dt + \int_0^1 8t+8-4t\, dt$

$-8\int_0^{\pi/2} \sin^2 t\, dt + 4\int_0^{\pi/2} \cos^2 t$ $\quad + \int_0^1 4t+8\,dt$

$2t^2 + 8t\Big|_0^1 = 2+8 = \boxed{10}$

↓

$-8\int_0^{\pi/2} \frac{1}{2} - \frac{\cos 2t}{2}\, dt$ $\quad 4\int_0^{\pi/2} \frac{1}{2} + \frac{\cos 2t}{2}\, dt$ $\quad \longrightarrow$

$-8\left[\frac{1}{2}t - \frac{\sin 2t}{4}\right]_0^{\pi/2}$ $\quad 4\left[\frac{1}{2}t + \frac{\sin 2t}{4}\right]_0^{\pi/2}$ $\quad 10 - 2\pi + \pi$

$= -8\,[(\pi/4 - 0) - (0)]$ $\quad 4\,[(\pi/4 + 0) - (0)]$ $\quad = \boxed{10-\pi}$ ✗

$= \boxed{-2\pi}$ $\quad \boxed{\pi}$

Problem 2. (10 points) Let $\mathbf{F} = \langle P, Q, R \rangle$ be an arbitrary vector field (where P, Q, R are continuous differentiable functions). Show that $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$.

$$F = \langle P, Q, R \rangle$$

$$\operatorname{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ P & Q & R \end{vmatrix} = \langle R_y - Q_z, -R_x + P_z, Q_x - P_y \rangle$$

$$\operatorname{div}(\operatorname{curl}(F)) = \langle R_y - Q_z, -R_x + P_z, Q_x - P_y \rangle \cdot \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$$

$$\operatorname{div}(\operatorname{curl}(F)) = \cancel{R_{yx}} - \cancel{Q_{xz}} - \cancel{R_{yx}} + P_{yz} + \cancel{Q_{xz}} - P_{yz}$$

$$\boxed{\operatorname{div}(\operatorname{curl}(F)) = 0}$$

which terms get cancelled?

9

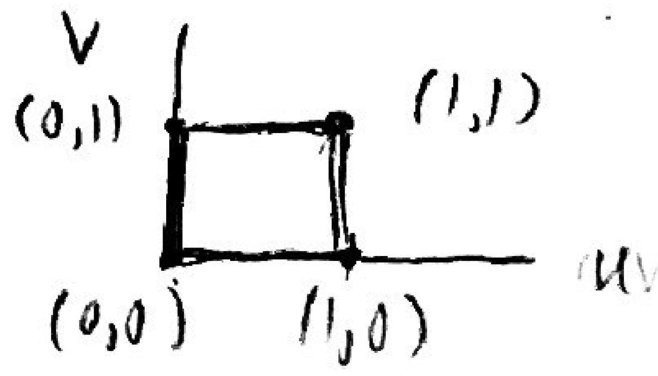
3 - 2 ?
· 3

Problem 1. (10 points) Let D be the parallelogram in the xy-plane with vertices $(0,0),(2,1),(1,2),(3,3)$.
(a) (5 points) Find a linear map $G(u,v)$ that maps the unit square $[0,1]\times[0,1]$ in the uv-plane to D in the xy-plane.

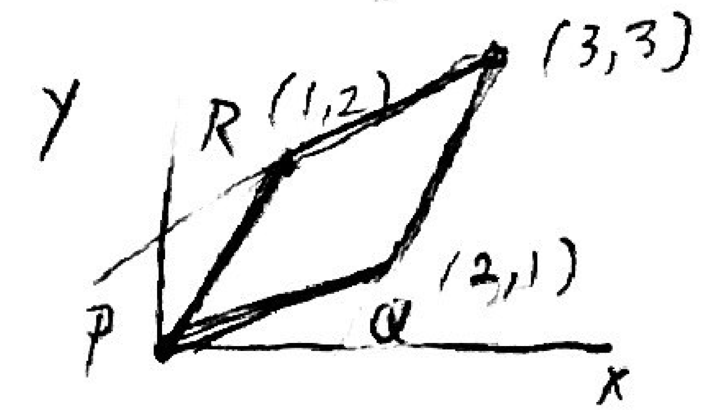
(b) (5 points) Evaluate $\iint_D (x+y)\,dx\,dy$ using change of variables given by $G(u,v)$.

$\frac{1}{2}x$

a)



vector PQ
A B
$(2,1)$
vector PR
C D
$(1,2)$



$3 = \frac{1}{2}(3) + c$

$3 - \frac{3}{2} = c$

$\frac{6}{2} - \frac{3}{2} = c$

$\frac{3}{2} = c$

$(Au + Cv,\ Bu + Dv)$
$\quad x \qquad y$
$(2u+v,\ u+2v)$

$y = \frac{1}{2}x \qquad y = \frac{1}{2}x + \frac{3}{2}$

$y = 2x \qquad y = 2x - 3$

$0 \le u \le 1 \qquad v = 0 \qquad u = 0$

$0 \le v \le 1 \qquad v = 1 \qquad u = 1$

$u \qquad v$
$2x+y,\ \ x+2y$

$\boxed{G(u,v) = (2u+v,\ u+2v) \quad [0,1]\times[0,1]}$

map ↑

u v	x y
$(0,0)$ =	$(0,0)$
$(1,0)$ =	$(2,1)$
$(1,1)$ =	$(3,3)$
$(0,1)$ =	$(1,2)$

$G_u = (2,1)$
$G_v = (1,2)$

$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 4 - 1 = \boxed{3}$

$\sqrt{3^2} = 3$ ✓

b) $\int_0^1\int_0^1 (2u+v+u+2v)\,|\mathrm{Jac}(G(u,v))|\,du\,dv$

$3\int_0^1\int_0^1 3u+3v\,du\,dv$

① $\int_0^1 3u+3v\,du$

$\frac{3u^2}{2} + 3uv \Big|_0^1 = 3\int_0^1 \frac{3}{2} + 3v\,dv$

$= 3\left(\frac{3}{2}v + \frac{3}{2}v^2 \Big|_0^1\right)$

$= 3\left(\frac{3}{2} + \frac{3}{2}\right)$

$= 3\left(\frac{6}{2}\right)$

$= \boxed{9}$

Math 32B-2 Yeliussizov. Midterm 2

Exam time: 5:00-6:30 PM, May 22, 2017

Last name: Rodriguez First name: Christian

Student ID: 804789345

Discussion: BOOZER 2A Tue, 2B Thu; KALYANSWAMY 2C Tue, 2D Thu; GUO 2E Tue, 2F Thu

There are 5 problems.
No books, notes, calculators, phones, conversations, etc.
Turn off your cell phones.

P 1 (10)	P 2 (10)	P 3 (15)	P 4 (10)	P 5 (15)	Total (60 pt)
~~8~~ 10	9	13	10	15	57