Midterm 2

Last Name:

First Name:

Student ID:

Signature:

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Zach Norwood
	1C	1D	TA: Eric Auld
	1E	1F	TA: Trent Hinkle

Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	10	10
2	10	~~10~~ 9 +1 EA
3	10	10
4	10	7
Total:	40	~~36~~ 37

3-D vector surface integral

1. (10 points) Consider the curve C parametrized by

$$\mathbf{r}(t) = (t-2)\mathbf{i} + (t^2 - 5)\mathbf{j} + \pi\sqrt{t-1}\mathbf{k}, \qquad 2 \le t \le 5.$$

$= \langle t-2, t^2-5, \pi\sqrt{t-1} \rangle$

Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ where

$$\mathbf{F}(x,y,z) = \left\langle y^2 + \frac{2xz}{1+x^2}, z\sin(yz) + 2xy, \ln(1+x^2) + y\sin(yz) \right\rangle.$$

$xy^2 + z\ln(1+x^2)$ $\quad$ $-\cos(yz) + xy^2$ $\quad$ $z\ln(1+x^2) - \cos(yz)$

potential function $f(x,y,z) = xy^2 + z\ln(1+x^2) - \cos(yz)$ ✓

partial derivatives match ✓

$\vec{F}$ is conservative ✓

$\int_C \vec{F}\cdot d\vec{r}$

$= f(\vec{r}(5)) - f(\vec{r}(2))$

$= f(5-2, 25-5, \pi\sqrt{5-1}) - f(2-2, 4-5, \pi\sqrt{2-1})$ ✓

$= f(3, 20, 2\pi) - f(0, -1, \pi)$

$= [3\cdot 400 + 2\pi\ln(1+9) - \cos(20\cdot 2\pi)] - [\cancel{0} + \cancel{\pi\ln(1)} - \cos(-\pi)]$

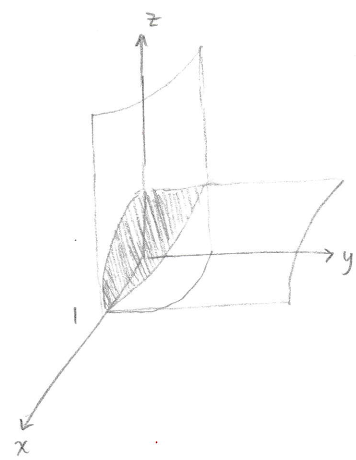
$\boxed{= [1200 + 2\pi\ln 10 - 1] - [-(-1)]}$ ✓

3-D scalar surface integral

2. (10 points) Let S be the portion of the cylinder $x^2 + z^2 = 1$ that is *inside* the cylinder $x^2 + y^2 = 1$ and where $x \geq 0$, $y \geq 0$, $z \geq 0$. Compute

$$\iint_S yz \, dS.$$

(Hint: Start with polar/cylindrical coordinates.)



$\vec{G}(\theta, y) = \langle \cos\theta, y, \sin\theta \rangle$

- $0 \leq \theta \leq \frac{\pi}{2}$
- $0 \leq y \leq \sqrt{1-x^2}$

$\hookrightarrow = \sqrt{1-\cos^2\theta} = \sin\theta$

- $\vec{T}_\theta = \langle -\sin\theta, 0, \cos\theta \rangle$
- $\vec{T}_y = \langle 0, 1, 0 \rangle$
- $\vec{N}(\theta, y) = \vec{T}_\theta \times \vec{T}_y$

$$= \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\sin\theta & 0 & \cos\theta \\ 0 & 1 & 0 \end{bmatrix}$$

$= \langle -\cos\theta, 0, -\sin\theta \rangle$ ✓

$\|\vec{N}(\theta, y)\| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\iint_S yz \, ds$

$= \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{1-\cos^2\theta}} y \sin\theta \cdot 1 \, dy \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin\theta \cdot \int_0^{\sqrt{1-\cos^2\theta}} y \, dy \, d\theta$

$= \int_0^{\frac{\pi}{2}} \sin\theta \left[\frac{1}{2} y^2\right]_0^{\sqrt{1-\cos^2\theta}} d\theta$

$= \frac{1}{2} \int_0^{\frac{\pi}{2}} \sin\theta \cdot (1-\cos^2\theta) \, d\theta$ Correct

$^* u := \cos\theta$

$du = -\sin\theta \, d\theta$

$-du = \sin\theta \, d\theta$

$= -\frac{1}{2} \int_1^0 (1-u^2) \, du$

$= \frac{1}{2} \left[u - \frac{1}{3} u^3\right]_0^1$

$$\boxed{= \frac{1}{2} \left[1 - \frac{1}{3}\right]}$$

9 + 1

EA

2-D vector line integral

3. (10 points) Suppose a magnetic field will impart on some object a force

$$\mathbf{F}(x, y, z) = \left\langle -ye^z, xe^z, \cancel{e^{x^2} - \cos(z^2)} \right\rangle$$

when $z=0$: $\vec{F}(x,y) = \langle -y, x \rangle$

when the object is at the point (x, y, z) in space. Compute the work done by this force in moving the object one full rotation counterclockwise (as viewed from above) around the ellipse

$$\frac{x^2}{4} + \frac{y^2}{16} = 1$$

call this C

in the xy-plane.

$\vec{r}(t) = \langle 2\cos t, 4\sin t \rangle$ ✓ $\quad 0 \le t \le 2\pi$ ✓

$d\vec{r} = \langle -2\sin t, 4\cos t \rangle \, dt$

$\int_C \vec{F} \cdot d\vec{r}$

$= \int_C \vec{F}(\vec{r}(t)) \cdot \langle -2\sin t, 4\cos t \rangle \, dt$

$= \int_0^{2\pi} \langle -4\sin t, 2\cos t \rangle \cdot \langle -2\sin t, 4\cos t \rangle \, dt$

$= \int_0^{2\pi} (8\sin^2 t + 8\cos^2 t) \, dt$

$= 8 \int_0^{2\pi} dt$ ✓

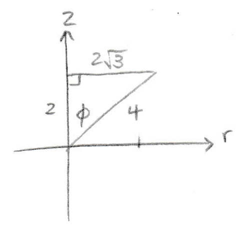
$\boxed{= 8[2\pi]}$

$= 16\pi$ ✓

3-D vector surface integral

4. (10 points) Let $\mathcal{S}$ be the portion of the sphere $x^2+y^2+z^2=16$ where $-2 \le z \le 2$ and $0 \le x \le y$. Compute the flux of the vector field $\langle yz, xz, xy\rangle$ through the surface $\mathcal{S}$, oriented with *inward* pointing normal vectors.

$\vec{G}(\theta,\phi) = \langle 4\sin\phi\cos\theta,\ 4\sin\phi\sin\theta,\ 4\cos\phi\rangle$



$* \ \underset{\frac{\pi}{4}}{\cancel{0}} \le \theta \le \underset{\frac{\pi}{2}}{\cancel{\frac{\pi}{4}}},\quad \frac{\pi}{3} \le \phi \le \frac{2\pi}{3}$

$\vec{T}_\theta = \langle -4\sin\phi\sin\theta,\ 4\sin\phi\cos\theta,\ 0\rangle$

$\vec{T}_\phi = \langle 4\cos\phi\cos\theta,\ 4\cos\phi\sin\theta,\ -4\sin\phi\rangle$

$\vec{N}(\theta,\phi) = \vec{T}_\theta \times \vec{T}_\phi$

$= \det\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4\sin\phi\sin\theta & 4\sin\phi\cos\theta & 0 \\ 4\cos\phi\cos\theta & 4\cos\phi\sin\theta & -4\sin\phi \end{bmatrix}$

$= \langle -16\sin^2\phi\cos\theta,\ -(16\sin^2\phi\sin\theta),\ -16\sin\phi\cos\phi\sin^2\theta - 16\sin\phi\cos\phi\cos^2\theta\rangle$

$= 16\sin\cancel{^2}\phi\,\langle -\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ -\cos\phi\rangle$ "inward-pointing normal vector" ✓

-1

$\hookrightarrow * \ \|\vec{N}(\theta,\phi)\| \overset{\vee}{=} \rho^2\sin\cancel{^2}\phi$

$\iint_S \langle yz, xz, xy\rangle\, d\vec{S}$

$= \int_{\cancel{0}\,\frac{\pi}{4}}^{\cancel{\frac{\pi}{4}}\,\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \langle 16\sin\phi\cos\phi\sin\theta,\ 16\sin\phi\cos\phi\sin\theta,\ 16\sin^2\phi\sin\theta\cos\theta\rangle$

-1

$\cdot\ 16\sin\cancel{^2}\phi\,\langle -\sin\phi\cos\theta,\ -\sin\phi\sin\theta,\ -\cos\phi\rangle\ d\phi\, d\theta$

-1 ⋮

7

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \langle 16 \sin\phi \cos\phi \sin\theta,\ 16 \sin\phi \cos\phi \sin\theta,\ 16 \sin^2\phi \sin\theta \cos\theta \rangle$$

$$\cdot\, 16 \sin\phi \,\langle -\sin\phi \cos\theta,\ -\sin\phi \sin\theta,\ -\cos\phi \rangle \, d\phi \, d\theta$$

$$= 256 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \cdot \left(-\sin^2\phi \cos\phi \,\underline{\cos^2\phi} - \sin^2\phi \cos\phi \,\underline{\sin^2\theta} \,\underline{- \sin^2\phi \cos\phi \sin\theta \cos\theta}\right) d\phi \, d\theta$$

$$= 256 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin\phi \left(\underline{-\sin^2\phi \cos\phi} \,\underline{- \sin^2\phi \cos\phi} \sin\theta \cos\theta\right) d\phi \, d\theta$$

$$= -256 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^3\phi \cos\phi \cdot (1 + \sin\theta \cos\theta) \, d\phi \, d\theta$$

$$= -256 \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta \right) \cdot \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin^3\phi \cos\phi \, d\phi$$

$*$ $u := \sin\phi$

$du = \cos\phi \, d\phi$

$$= -256 \left(\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin\theta \cos\theta \, d\theta \right) \cdot \cancel{\int_{\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} u^3 \, du}$$

$$= \boxed{0}$$