Midterm 1

Last Name: ______________________

First Name: ______________________

Student ID: ______________________

Signature: ______________________

Section:

	Tuesday:	Thursday:	
	1A	1B	TA: Zach Norwood
	1C	1D	TA: Eric Auld
	1E	1F	TA: Trent Hinkle

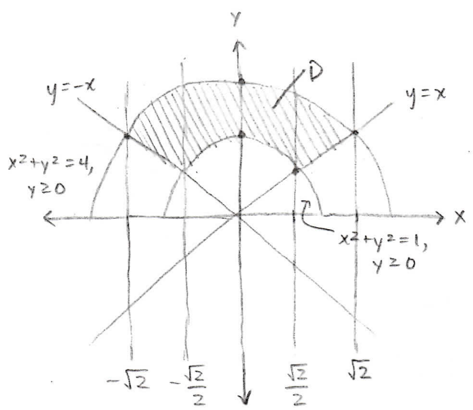
Instructions: Do not open this exam until instructed to do so. You will have 50 minutes to complete the exam. Please print your name and student ID number above, and circle the number of your discussion section. **You may not use calculators,** books, notes, or any other material to help you. Please make sure **your phone is silenced** and stowed where you cannot see it. You may use any available space on the exam for scratch work. If you need more scratch paper, please ask one of the proctors. You must **show your work** to receive credit. Please circle or box your final answers.

Please do not write below this line.

Question	Points	Score
1	10	10
2	10	10
3	10	6
4	10	9
Total:	40	35

1. (10 points) Compute the following, in any way you like. *(Hint: Draw a picture!)*

$$\int_{\frac{\sqrt{2}}{2}}^{\sqrt{2}} \int_{x}^{\sqrt{4-x^2}} y^2\, dy\, dx + \int_{-\frac{\sqrt{2}}{2}}^{\frac{\sqrt{2}}{2}} \int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} y^2\, dy\, dx + \int_{-\sqrt{2}}^{-\frac{\sqrt{2}}{2}} \int_{-x}^{\sqrt{4-x^2}} y^2\, dy\, dx$$



$y \le \sqrt{4-x^2}$

$y^2 \le 4-x^2, \; y \ge 0$

$x^2+y^2 \le 4, \; y \ge 0$

$y \ge \sqrt{1-x^2}$

$y^2 \ge 1-x^2, \; y \ge 0$

$x^2+y^2 \ge 1, \; y \ge 0$

$\iint_D y^2 \, dA$

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \int_1^2 (r\sin\theta)^2 \, r\, dr\, d\theta$ ✓

$= \int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \sin^2\theta \, d\theta \cdot \int_1^2 r^3\, dr$

$= \frac{1}{4}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (2-2\cos 2\theta)\, d\theta \cdot \int_1^2 r^3\, dr$ ✓

$= \frac{1}{4}\left[2\theta - \sin 2\theta\right]_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \cdot \left[\frac{1}{4}r^4\right]_1^2$

$= \frac{1}{4}\left[\left(\frac{3\pi}{2} - \sin\frac{3\pi}{2}\right) - \left(\frac{\pi}{2} - \sin\frac{\pi}{2}\right)\right] \cdot \frac{1}{4}\left[2^4 - 1\right]$

$= \frac{1}{4}\left[\left(\frac{3\pi}{2} - (-1)\right) - \left(\frac{\pi}{2} - 1\right)\right] \cdot \frac{1}{4}\left[2^4 - 1\right]$

$= \frac{1}{4}\left[\left(\frac{3\pi}{2} + 1\right) - \left(\frac{\pi}{2} - 1\right)\right) \cdot \frac{1}{4}(15)$

$= \frac{1}{4}(\pi + 2) \cdot \frac{15}{4}$ ✓

$= \frac{15}{16}(\pi + 2)$

10

2. (10 points) A space station is designed as a sphere of radius 2 km with a cylinder of radius 1 km removed from the center of it. The heat density in the air in the space station is given by

$$\frac{1}{x^2+y^2+z^2},$$

where x, y, and z are measured in km, with $(0,0,0)$ located at the center of the sphere. Compute the total amount of heat in the air inside the space station.

$$\iiint_W \frac{1}{x^2+y^2+z^2}\,dV$$

$$= \int_0^{2\pi}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_{\csc\phi}^{2} \frac{1}{\cancel{\rho^2}}\,\cancel{\rho^2}\,d\rho\,\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\phi \int_{\csc\phi}^{2} d\rho\,d\phi$$

$$= [\theta]_0^{2\pi} \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\phi \cdot [\rho]_{\csc\phi}^{2}\,d\phi$$

$$= [2\pi] \cdot \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\phi \cdot [2-\csc\phi]\,d\phi$$

$$= 2\pi \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} [2\sin\phi - 1]\,d\phi$$

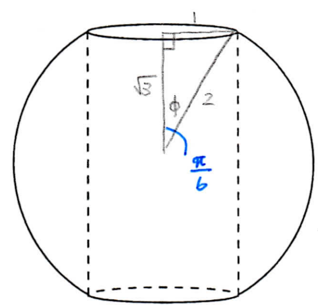
$$= 2\pi\,[-2\cos\phi - \phi]_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

$$= 2\pi\left[\left(-2\cos\tfrac{5\pi}{6} - \tfrac{5\pi}{6}\right) - \left(-2\cos\tfrac{\pi}{6} - \tfrac{\pi}{6}\right)\right]$$

$$= 2\pi\left[\left(-2\cdot\left(-\tfrac{\sqrt{3}}{2}\right) - \tfrac{5\pi}{6}\right) - \left(-2\left(\tfrac{\sqrt{3}}{2}\right) - \tfrac{\pi}{6}\right)\right]$$

$$= 2\pi\left[\left(\sqrt{3} - \tfrac{5\pi}{6}\right) - \left(-\sqrt{3} - \tfrac{\pi}{6}\right)\right]$$

$$= 2\pi\left[2\sqrt{3} - \tfrac{2\pi}{3}\right]$$



$x^2+y^2 \geq 1$

$r^2 \geq 1$

$(\rho\sin\phi)^2 \geq 1$

$\rho\sin\phi \geq 1$

$\rho \geq \csc\phi$

3. (10 points) Let $\mathcal{W}$ be the region in the first octant above $z = x^2 + y^2$ and below $z = 2\sqrt{x^2+y^2}$. Compute

$$\iiint_{\mathcal{W}} x \, dV.$$

$$\int_0^{2\pi} \int_0^2 \int_{2r}^{r^2} r\cos\theta \, dz \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} \cos\theta \, d\theta \cdot \int_0^2 r^2 \int_{2r}^{r^2} dz \, dr$$

$$= [\sin\theta]_0^{2\pi} \cdot$$

$$= 0$$

~~10~~

6/10

assumed symmetry

didn't integrate

$$\int_0^{\pi/2}$$

- $z \geq x^2 + y^2$

$z \geq r^2$

(paraboloid)

- $z \leq 2\sqrt{x^2+y^2}$

$z \leq 2r$

(cone)

- $r^2 \leq 2r$

$r \leq 2$

(intersection)

$$\int_0^{\frac{\pi}{2}} \int_0^2 \int_{r^2}^{2r} r\cos\theta \, dz \, r \, dr \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta \cdot \int_0^2 r^2 \int_{r^2}^{2r} dz \, dr$$

$$= [\sin\theta]_0^{\frac{\pi}{2}} \cdot \int_0^2 r^2 \cdot [2r - r^2] \, dr$$

$$= [1] \cdot \int_0^2 [2r^3 - r^4] \, dr$$

$$= \left[\frac{1}{2} r^4 - \frac{1}{5} r^5\right]_0^2$$

$$= \frac{1}{2} \cdot 16 - \frac{1}{5} \cdot 32$$

$$= \frac{40}{5} - \frac{32}{5}$$

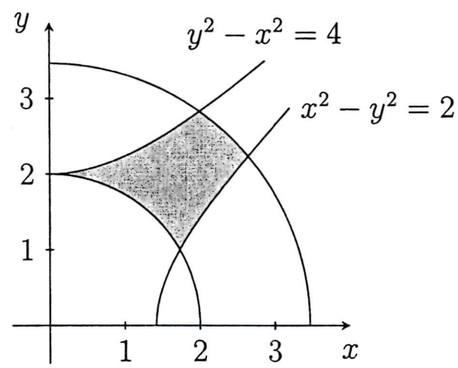
$$\boxed{= \frac{8}{5}}$$

4. (10 points) Let $\mathcal{D}$ be the region in the first quadrant bounded by the circles $x^2+y^2=4$ and $x^2+y^2=12$, and the hyperbolae $x^2-y^2=2$ and $y^2-x^2=4$. Use the change of variables

$$G(u,v) := \begin{cases} x=\sqrt{u+v} \\ y=\sqrt{u-v} \end{cases}$$

to compute the integral

$$\iint_{\mathcal{D}} 2xy^3\,dA.$$



new bounds:

- $x^2+y^2=4$
  $(u+v)+(u-v)=4$
  $2u=4$
  $u=2$
- $x^2+y^2=12$
  $(u+v)+(u-v)=12$
  $2u=12$
  $u=6$
- $x^2-y^2=2$
  $(u+v)-(u-v)=2$
  $2v=2$
  $v=1$
- $y^2-x^2=4$
  $(u-v)-(u+v)=4$
  $-2v=4$
  $v=-2$

$$\det \text{Jac } G = \det \begin{bmatrix} \frac{1}{2\sqrt{u+v}} & \frac{1}{2\sqrt{u+v}} \\ \frac{1}{2\sqrt{u-v}} & \frac{-1}{2\sqrt{u-v}} \end{bmatrix}$$

$$= \frac{-1}{4\sqrt{u+v}\sqrt{u-v}} - \frac{1}{4\sqrt{u+v}\sqrt{u-v}}$$

$$= -\frac{1}{2\sqrt{u+v}\sqrt{u-v}}$$

abs $\to \dfrac{1}{2\sqrt{u+v}\sqrt{u-v}}$

$$\int_{-2}^{1}\int_{2}^{6} 2\sqrt{u+v}\sqrt{u-v}^{\,3}\cdot\frac{1}{2\sqrt{u+v}\sqrt{u-v}}\,du\,dv$$

$$= \int_{-2}^{1}\int_{2}^{6}(u-v)\,du\,dv = 54$$

$$= \int_{-2}^{1}\left[\tfrac{1}{2}u^2 - vu\right]_2^6 dv$$

$$= \int_{-2}^{1}\left[\tfrac{1}{2}36 - 6v\right] - \left[\tfrac{1}{2}4 - 2v\right] dv$$

$$= \int_{-2}^{1}(16-4v)\,dv$$

$$= \left[16v - 2v^2\right]_{-2}^{1} \Longrightarrow [16-2]-[16(-2)-2(4)]$$

$$= 14 - [-32-8]$$

$$\boxed{=54}$$

$$= [16-2]-[-32-4]$$

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