Math 32B - Spring 2019

Exam 2

Full Name: Sadhana Vadrevu

UID: 205095030

Circle the name of your TA and the day of your discussion:

Patrick Hiatt   Eli Sadovnik   Frederick Vu

Tuesday   Thursday

Instructions:

- Read each problem carefully.
- Show all work clearly and circle or box your final answer where appropriate.
- Justify your answers. A correct final answer without valid reasoning will not receive credit.
- Simplify your answers as much as possible.
- Include units with your answer where applicable.
- Calculators are not allowed but you may have a 3 $\times$ 5 inch notecard.

Page	Points	Score
1	15	
2	30	
3	30	
4	25	
Total:	100	

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You may use this page for scratch work. Work found on this page will not be graded unless clearly indicated in the exam.

1. (15 points) Let $\mathbf{F}(x,y,z) = \langle yz, xz, xy{+}2z \rangle$ and let C be the line segment from $(1,0,-2)$ to $(4,5,3)$.

(a) Show that the vector field $\mathbf{F}$ is conservative using curl.

$$\operatorname{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy+2z \end{vmatrix} = \langle x-x,\ y-y,\ z-z \rangle = \langle 0,0,0 \rangle = \vec{0}$$

$\operatorname{curl}(\vec{F}) = \vec{0} \therefore \vec{F}$ is conservative ✓

(b) Find a function f such that $\mathbf{F} = \nabla f$.

$\int F_1\,dx = yzx + C_x$

$\int F_2\,dy = xzy + C_y$

$\int F_3\,dy = xyz + z^2 + C_z$

$\therefore \boxed{f = xyz + z^2}$

(c) Use part (b) to evaluate $\displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$.

$\vec{F}$ is conservative, so can use Fundamental Theorem of Line Integrals

$\int_C \vec{F} \cdot d\vec{r} = f(4,5,3) - f(1,0,-2) = 69 - 4 = \boxed{65}$

$f(4,5,3) = 4 \cdot 5 \cdot 3 + 3^2 = 60 + 9 = 69 \quad f(1,0,-2) = 1 \cdot 0 \cdot -2 + (-2)^2 = 4$

(d) Is there a vector field $\mathbf{G}$ defined on $\mathbb{R}^3$ such that $\operatorname{curl}\mathbf{G} = \mathbf{F}$? $\boxed{\text{no}}$

$\operatorname{div}(\vec{F}) = 0 + 0 + 2 = 2 \neq 0 \therefore$

there is no $\vec{G}$ that exists such that $\operatorname{curl}(\vec{G}) = \vec{F}$

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2. (15 points) Consider the vector field

$$\mathbf{F}(x,y) = \langle F_1, F_2 \rangle = \left\langle \frac{y}{x^2-y^2}, \frac{-x}{x^2-y^2} \right\rangle.$$

(a) Show that $\dfrac{\partial F_1}{\partial y} = \dfrac{\partial F_2}{\partial x}$.

$$\frac{\partial F_1}{\partial y} = \frac{(x^2-y^2) - y(-2y)}{(x^2-y^2)^2} = \frac{x^2-y^2+2y^2}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$\frac{\partial F_2}{\partial x} = \frac{(x^2-y^2)(-1) - (-x)(2x)}{(x^2-y^2)^2} = \frac{-x^2+y^2+2x^2}{(x^2-y^2)^2} = \frac{x^2+y^2}{(x^2-y^2)^2}$$

$$\therefore \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x} = \frac{x^2+y^2}{(x^2-y^2)^2} \checkmark$$

(b) Show that $\mathbf{F}$ is defined on four distinct connected domains in the plane. On each of these domains, is $\mathbf{F}$ conservative? *Hint:* Are these domains simply connected?

$\vec{F}$ is not defined @ $(0,0)$ but is defined everywhere else

simply connected $\therefore \vec{F}$ is conservative $\Leftarrow \{(x,y) \mid x<0, y>0\}$ D_2

D_1: $\{(x,y) \mid x>0, y>0\} \Rightarrow$ simply connected $\therefore \vec{F}$ is conservative

simply connected $\therefore \vec{F}$ is conservative $\Leftarrow \{(x,y) \mid x<0, y<0\}$: D_3

D_4 $\{(x,y) \mid x>0, y<0\} \Rightarrow$ simply connected $\therefore \vec{F}$ is conservative

$\boxed{\vec{F} \text{ is conservative on each of the 4 domains}}$

3. (15 points) Find the work done by the force field $\mathbf{F}(x,y) = \langle x^2, ye^x \rangle$ in moving a particle along the parabola $x = y^2+1$ from $(1,0)$ to $(2,1)$.

$\vec{r}(t) = \langle t^2+1, t \rangle \quad 0 \le t \le 1 \qquad \vec{r}\,'(t) = \langle 2t, 1 \rangle$

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 \langle t^4+2t^2+1, te^{t^2+1} \rangle \cdot \langle 2t, 1 \rangle \, dt$$

$$= \int_0^1 2t^5 + 4t^3 + 2t + te^{t^2+1} \, dt$$

$$= \frac{1}{3}t^6 + t^4 + t^2 + \frac{1}{2}e^{t^2+1} \Big|_0^1$$

$$= \frac{1}{3} + 1 + 1 + \frac{1}{2}e^2 - \frac{1}{2}e = \boxed{\frac{7}{3} + \frac{1}{2}e^2 - \frac{1}{2}e}$$

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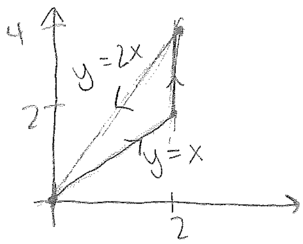
4. (15 points) Use Green's Theorem to evaluate the line integral $\int_C xy^2\,dx + 2x^2y\,dy$ along the positively oriented boundary C of the triangle with vertices $(0,0)$, $(2,2)$, and $(2,4)$.



$\int_C xy^2\,dx + 2x^2y\,dy$.

$F_1 = xy^2 \Rightarrow F_{1y} = 2xy \quad F_2 = 2x^2y \Rightarrow F_{2x} = 4xy$.

$F_{2x} - F_{1y} = 2xy$

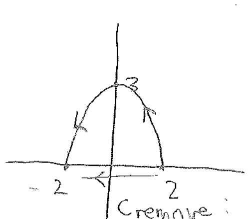
$\int_C xy^2\,dx + 2x^2y\,dy \overset{\text{GREENS}}{=} \iint_D 2xy\,dA$

$= \int_0^2 \int_x^{2x} 2xy\,dy\,dx = \int_0^2 xy^2\Big|_x^{2x}\,dx$

$= \int_0^2 x(4x^2) - x(x^2)\,dx = \int_0^2 3x^3\,dx = \frac{3}{4}x^4\Big|_0^2$

$= 3(4) = \boxed{12}$

5. (15 points) Let C be the curve parameterized by $\mathbf{r}(t) = \langle 2\cos t, 3\sin t\rangle$ for $0 \le t \le \pi$. Find $\int_C (e^x + y^2\cos x)\,dx + (e^{2y} + 2y\sin x)\,dy$. *Hint:* Complete C to form a closed curve and use Green's Theorem.



Cremove:

$\vec{r}(t) = \langle t, 0\rangle$

$2 \ge t \ge -2$

$\int_C (e^x + y^2\cos x)\,dx + (e^{2y} + 2y\sin x)\,dy$

$= \oint_C \underbrace{(e^x + y^2\cos x)}_{F_1 \Rightarrow F_{1y} = 2y\cos x}\,dx + \underbrace{(e^{2y} + 2y\sin x)}_{F_{2x} = 2y\cos x}\,dy + \int_{C_{remove}} \vec{F}\cdot d\vec{r}$

$F_{2x} - F_{1y} = 0$

$\oint_C (e^x + y^2\cos x)\,dx + (e^{2y} + 2y\sin x)\,dy \overset{\text{GREENS}}{=} \iint_D 0\,dA = 0$

$\int_{C_{remove}} \vec{F}\cdot d\vec{r} = \int_2^{-2} \langle e^t, 1\rangle \cdot \langle 1, 0\rangle\,dt = \int_2^{-2} e^t\,dt = e^t\Big|_2^{-2}$

$= \frac{1}{e^2} - e^2$

$\therefore \int_C (e^x + y^2\cos x)\,dx + (e^{2y} + 2y\sin x)\,dy = \boxed{\frac{1}{e^2} - e^2}$

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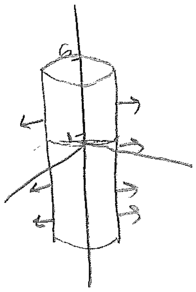
6. (25 points) Consider the vector field $\mathbf{F}(x,y,z) = \langle x, y, e^z \rangle$ and the oriented surface $\mathcal{S}$ given by the cylinder $x^2+y^2=4$ for $1 \le z \le 6$ with outward pointing normal. (Note: the surface $\mathcal{S}$ does not include the top or bottom of the cylinder).

(a) Find the flux $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$.

$S: \vec{r}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, z\rangle \quad 0 \le \theta \le 2\pi, \ 1 \le z \le 6$



$r_z = \langle 0, 0, 1\rangle \qquad \vec{n} = \vec{r}_\theta \times \vec{r}_z = \langle 2\cos\theta, 2\sin\theta, 0\rangle$

$r_\theta = \langle -2\sin\theta, 2\cos\theta, 0\rangle$

$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(\theta, z)) \cdot (\vec{r}_\theta \times \vec{r}_z)\, dA$

$= \int_0^{2\pi} \int_1^6 \langle 2\cos\theta, 2\sin\theta, e^z\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle\, dz\, d\theta$

$= \int_0^{2\pi} \int_1^6 4\cos^2\theta + 4\sin^2\theta\, dz\, d\theta$

$= 4\int_0^{2\pi} \int_1^6 dz\, d\theta = 4(2\pi)(5) = \boxed{40\pi}$

(b) Use a surface integral to compute the surface area of $\mathcal{S}$. (You should be able to check your answer by computing the surface area another way).

$SA = \iint_S 1\, dS = \iint_D \|\vec{r}_\theta \times \vec{r}_z\|\, dz\, d\theta = \int_0^{2\pi}\int_1^6 2\, dz\, d\theta$

$\|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{4} = 2 \qquad = 2(2\pi)(5) = \boxed{20\pi}$

check: $SA = 2\pi r \times h = 2\pi(2)(6-1) = 2\pi(2)(5) = 20\pi$ ✓

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